Calculus II: Partial Fraction Decomposition

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1. **Recall u-sub, especially ln—x—**
2. **Write integrals on the board all du/u**
3. **Explain P(x)/ Q(x) terms**
4. **Table of factors and terms**
5. **Process**
   (a) *Identify P(x)/ Q(x) check degP<degQ*
   (b) *Factor Q into easy parts*
   (c) *Find Ai's for it*
   (d) *Use U-sub for it*

1 Introduction

Solve

$$\int \frac{2x-1}{x^2-x-6}\,\mathrm{d}x\,. \tag{1}$$

Using "u-substitution" where $u = x^2 - x - 6$ and $\mathrm{d}u = 2x - 1\,\mathrm{d}x$, we see

$$\begin{aligned}\int \frac{2x-1}{x^2-x-6}\,\mathrm{d}x &\overset{\mathrm{u=x^2-x-6}}{\Longrightarrow} \int \frac{\mathrm{d}u}{u}\\ &= \ln|u|\\ &= \ln\left|x^2-x-6\right|\end{aligned}$$

But what if we can't find a way to write the numerator in terms of the denominator, like

$$\int \frac{3x+11}{x^2-x-6}\,\mathrm{d}x \qquad \text{or} \qquad \int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}\,\mathrm{d}x$$

2 Partial Fraction Decomposition

The idea of partial fraction decomposition (PFD) is to rewrite the integrand in terms of simpler factors that we know how to integrate. This process works when you have a two polynomial functions $P(x)$ in the numerator and $Q(x)$ in the denominator and you find a way to factor Q.

2.1 The Process

1. *Identify $P(x)$ and $Q(x)$ in integrand. Then check* $\deg P < \deg Q$[1].
2. *Factor Q into easy parts. See what terms are needed based on the table below.*

Factor in denominator	Term in PFD
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^n$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_3}{(ax+b)^n}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^n$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$

3. *Find A_i, B_i's for it by making a common denominator.*
4. *Integrate the factors*

Example 2.1. *Using PFD, solve*

$$\int \frac{2x-1}{x^2-x-6}\,\mathrm{d}x\,. \tag{2}$$

Solution. We use the steps outlined in the previous part.

1. Notice how $P(x) = 2x-1$ and $Q(x) = x^2-x-6$, with $\deg P = 2 < 3 = \deg Q$.
2. Notice how $Q(x) = x^2-x-6 = (x-3)(x+2)$. This means we have two terms in the PFD

$$(x-3) \Rightarrow \frac{A_1}{x-3}$$
$$(x+2) \Rightarrow \frac{A_2}{x+2}.$$

3. We now expect that we can write the integrand using these factors

$$\frac{2x-1}{x^2-x-6} = \frac{A_1}{x-3} + \frac{A_2}{x+2}$$
$$\frac{2x-1}{x^2-x-6} = \frac{A_1(x+2)+A_2(x-3)}{(x-3)(x+2)}.$$

They now have a common denominator, so we know $2x-1 = A_1(x+2)+A_2(x-3)$, so we have a problem with two unknowns, A_1 and A_2 and two equations.

Wait! Where is the second equation? We need the coefficients in the x term to be equal and the constant term to be equal, so those are thetwo equations.

$$2 = A_1 + A_2$$
$$-1 = 2A_1 - 3A_2$$

Using your favorite solution technique, we can show $A_1 = 1, A_2 = 1$, so the integrand becomes

$$\frac{2x-1}{x^2-x-6} = \frac{1}{x-3} + \frac{1}{x+2}.$$

[1] The PFD technique works even when $\deg P \geqslant \deg Q$. One first has to divide P by Q to get the quotient and the remainder. Then we integrate the quotient and can apply PFD to the remainder. However, this is beyond the scope of this mini-lecture.

4. Now that we have rewritten the integrand, we have to solve the integral.

$$\begin{aligned}\int \frac{2x-1}{x^2-x-6}\,\mathrm{d}x &= \int \frac{1}{x-3} + \frac{1}{x+2}\,\mathrm{d}x \\ &= \int \frac{1}{x-3}\,\mathrm{d}x + \int \frac{1}{x+2}\,\mathrm{d}x \\ &= \ln|x-3| + \ln|x+2| \\ &= \ln|(x-3)(x+2)| \\ &= \ln\left|x^2-x-6\right|,\end{aligned}$$

as expected.

Example 2.2. *Solve*

$$\int \frac{x^2}{(x-2)(x-3)^2}\,\mathrm{d}x\,. \tag{3}$$

Solution. Since the derivative of the denominator will have many terms, we cannot hope that the $Q(x)$ has any relation to $P(x)$, so "u-sub" will not work. Let's try to do PFD, so we use the steps outlined in the previous part.

1. Notice how $P(x) = x^2$ and $Q(x) = (x-2)(x-3)^2$, with $\deg P = 2 < 3 = \deg Q$.
2. Notice how $Q(x) = (x-2)(x-3)^2$, which is already factored into "nice" polynomial and cannot be further broken down.

$$\begin{aligned}(x-2) &\Rightarrow \frac{A_1}{x-2} \\ (x-3)^2 &\Rightarrow \frac{A_2}{x-3} + \frac{A_3}{(x-3)^2}.\end{aligned}$$

3. We now expect that we can write the integrand using these factors.

$$\begin{aligned}\frac{x^2}{(x-2)(x-3)^2} &= \frac{A_1}{x-2} + \frac{A_2}{x-3} + \frac{A_3}{(x-3)^2} \\ &= \frac{A_1(x-3)^2 + A_2(x-2)(x-3) + A_3(x-2)}{(x-2)(x-3)^2}\end{aligned}$$

Since the denominators are the same, the only way we can hope for equality is if the numerators equal.

$$\begin{aligned}x^2 &= A_1(x-3)^2 + A_2(x-2)(x-3) + A_3(x-2) \\ &= A_1x^2 + 6A_1x + 9A_1 + A_2x^2 - 5A_2x + 6A_2 + A_3x - 2A_3 \\ &= (A_1 + A_2)x^2 + (-6A_1 - 5A_2 + A_3)x + (9A_1 + 6A_2 - 2A_3)\end{aligned}$$

Since the coefficents for each degree of x have to be the same, we get the following three equations and three unknowns.

$$\begin{aligned}A_1 + A_2 \qquad &= 1 \\ -6A_1 - 5A_2 + A_3 &= 0 \\ 9A_1 + 6A_2 - 2A_3 &= 0\end{aligned}$$

Solving this system using Gaussian Elimination, we get $A_1 = 4, A_2 = -3, A_3 = 9$, so

$$\frac{x^2}{(x-2)(x-3)^2} = \frac{4}{x-2} + \frac{-3}{x-3} + \frac{9}{(x-3)^2}$$

4. Now that we have rewritten the integrand, we have to solve the integral.

$$\begin{aligned}\int \frac{x^2}{(x-2)(x-3)^2}\,\mathrm{d}x &= \int \frac{4}{x-2} + \frac{-3}{x-3} + \frac{9}{(x-3)^2}\,\mathrm{d}x \\ &= \int \frac{4}{x-2}\,\mathrm{d}x + \int \frac{-3}{x-3}\,\mathrm{d}x + \int \frac{9}{(x-3)^2}\,\mathrm{d}x \\ &= 4\ln|x-2| - 3\ln|x-3| - \frac{9}{x-3}\end{aligned}$$