# Emory University
# Practice Final Exam
# MA-210 Advanced Data Science Calculus

Date: May 1, 2025

Instructor: Head TA Mitchell Scott

Student ID ____________ Name: ____________________________

**Please read the following instructions carefully.**

- ***This is simpliy a review session!*** This question booklet contains 6 questions, 7 pages (including the cover) for the total of 95 points/marks. Check to see if any pages are missing. DO NOT scribble or do rough work or make any stray marks on it. Use separate sheet for rough work.
- This is meant to represent what an actual exam might look like. **Read the instructions for individual questions carefully** before answering the questions.
- No instructor was consulted for the making of this assignment.

| Question | Points | Score |
| --- | --- | --- |
| 1 | 5 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 28 | |
| 5 | 22 | |
| 6 | 20 | |
| Total: | 95 | |

1. (5 points) **Exponential variables can also be used to model situations where certain events occur with a constant probability per unit length, such as the distance between mutations on a DNA strand, or between roadkills on a given road. Let $X$ be a random variable pulled from this exponential distribution, where the pdf is**

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \textbf{elsewhere}. \end{cases}$$

**What is the $\mathbb{E}[X]$? (n.b. $\lambda > 0$ is what is called the rate parameter, but you can think of it as a constant.)**

**Solution:** First, we recall that

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} x \cdot f_X(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{0} x \cdot 0\,\mathrm{d}x + \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\,\mathrm{d}x\,. \end{aligned}$$

The first term is clearly zero, and the second term requires integration by parts. Using ILATE, we choose $u = x, \mathrm{d}v = \lambda e^{-\lambda x}$.

$$\begin{aligned} \mathbb{E}[X] &= \int_0^{\infty} x\lambda e^{-\lambda x}\,\mathrm{d}x \\ &= -x\lambda e^{-\lambda x}\Big|_0^{\infty} - \int_0^{\infty} -e^{-\lambda x}\,\mathrm{d}x \\ &= (-0+0) - \frac{1}{\lambda e^{-\lambda x}}\Big|_0^{\infty} \\ &= -\frac{1}{\lambda e^{\lambda\infty}} + \frac{1}{\lambda e^{\lambda\cdot 0}} \\ &= \frac{1}{\lambda} \end{aligned}$$

2. (10 points) **Find the volume of the solid that lies below the function surface given by $f(x,y) = 6x^2y + 20x$ and lies above the region in the $xy$-plane bounded by $y = x^2$ and $y = 2 - x^2$.**

**Solution:** First, we need to find the bounds

$$x^2 = 2 - x^2 \implies 2x^2 = 2 \implies x = \pm 1$$

so $-1 \leq x \leq 1$, and $x^2 \leq y \leq 2 - y^2$. This means we get the integral

$$\begin{aligned} V &= \iint_D 6x^2y + 20x \, \mathrm{d}A \\ &= \int_{-1}^{1} \int_{x^2}^{2-x^2} 6x^2y + 20x \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_{-1}^{1} \left( 3x^2y^2 + 20xy \Big|_{x^2}^{2-x^2} \right) \mathrm{d}x \\ &= \int_{-1}^{1} 3x^2(2-x^2)^2 + 20x(2-x^2) - 3x^2x^4 - 20xx^2 \, \mathrm{d}x \\ &= \int_{-1}^{1} 12x^2 - 12x^4 + 40x - 40x^3 \, \mathrm{d}x \\ &= 4x^3 - \frac{12x^5}{5} + 20x^2 - 10x^4 \Big|_{-1}^{1} \\ &= 8 - \frac{24}{5} \\ &= \frac{16}{5} \end{aligned}$$

3. (10 points) **Let $f(x, y) = 4x^2 + 3y^2$. Suppose $x$ and $y$ are functions of $t$ such that**

$$x(t) = \sin t$$
$$y(t) = \cos t$$

**Compute $\frac{\mathrm{d}f}{\mathrm{d}t}$ at $t = 3\pi/4$.**

**Solution:** Recall the multidimensional chain rule

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

Taking it piece by piece we have

$$\frac{\partial f}{\partial x} = 8x$$
$$\frac{\partial x}{\partial t} = \cos t$$
$$\frac{\partial f}{\partial y} = 6y$$
$$\frac{\partial y}{\partial t} = -\sin t$$

This means that

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} \\
&= (8x)(\cos t)+(6y)(-\sin t) \\
\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=3\pi/4} &= 8x(3\pi/4)\left(\frac{-\sqrt{2}}{2}\right) - 6y(3\pi/4)\left(\frac{-\sqrt{2}}{2}\right) \\
&= 8\frac{\sqrt{2}}{2}\frac{-\sqrt{2}}{2} - 6\frac{-\sqrt{2}}{2}\frac{-\sqrt{2}}{2} \\
&= -4-3=-7
\end{aligned}$$

4. (14 points) **President Fenvez has a paperweight on his desk with density** $xz$**. The paperweight is a tetrahedron with corners** $(0,0,0),(0,1,0),(1,1,0),(0,1,1)$**.**

   (a) (7 points) **Set up a triple integral with differential** $\mathrm{d}V = \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ **that represents the mass of the paperweight** $P$**.**

   **Solution:** The dimensions of the paperweight are

   $$P = \{(x,y,z)\in\mathbb{R}^3 | 0\le x\le 1, x\le y\le 1, 0\le z\le y-x\}$$

   Looking at the bounds, we are quickly able to see

   $$M = \int_0^1\int_x^1\int_0^{y-x} xz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

   (b) (7 points) **Solve the integral from part (a) to compute the mass of the paperweight.**

   **Solution:**

   $$\begin{aligned}
   M &= \int_0^1\int_x^1\int_0^{y-x} xz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
   &= \int_0^1\int_x^1 \frac{x(y-x)^2}{2}\,\mathrm{d}y\,\mathrm{d}x \\
   &= \frac{1}{2}\int_0^1 \frac{x(1-x)^3}{3}\,\mathrm{d}x \\
   &= \frac{1}{6}\int_0^1 x-3x^2+3x^2-x^4\,\mathrm{d}x \\
   &= \frac{1}{120}
   \end{aligned}$$

5. (11 points) **Let $X$ and $Y$ be two jointly continuous random variables with joint PDF**

$$f_{XY}(x,y) = \begin{cases} x + cy^2, & 0 \leq x, y \leq 1 \\ 0, & \textbf{elsewhere} \end{cases}$$

   **(a) (4 points) Set up the integral representing the probability**

**Solution:**

$$\begin{aligned} \mathbb{P}[(X,Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \int_0^1 f_{XY}(x,y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

   **(b) (7 points) Solve for $c$.**

**Solution: To do this, we have to remember that $\mathbb{P}(x,y) = 1$, so**

$$\begin{aligned} 1 &= \int_0^1 \int_0^1 f_{XY}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \int_0^1 x + cy^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \left( \frac{1}{2}x^2 + cy^2 x \Big|_{x=0}^{x=1} \right) \mathrm{d}y \\ &= \int_0^1 \frac{1}{2} + cy^2\,\mathrm{d}y \\ &= \frac{1}{2}y + \frac{1}{3}cy^3 \Big|_{y=0}^{y=1} \\ &= \frac{1}{2} + \frac{c}{3}. \end{aligned}$$

**Solving for $c$, we get $c = \frac{3}{2}$**

6. **Let $f(x) = e^{-3x}$.**

   (a) (10 points) **Write a Taylor Series for $f$ centered at $x = -2$.**

**Solution:** Recall that the formula for a Taylor series is

$$T(x;a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{1}$$

Here it is easy to see that this term is centered around $a = -2$. This means we need to find $f^{(n)}(a)$.

$$\begin{aligned} f(x) &= e^{-3x} \\ f'(x) &= (-3)e^{-3x} \\ f''(x) &= (-3)^2 e^{-3x} \\ f^{(3)}(x) &= (-3)^3 e^{-3x} \\ &\vdots \end{aligned}$$

Clearly, we see that $f^{(n)}(x) = (-3)^n e^{-3x} \implies f^{(n)}(a) = (-3)^n e^{-3a} \implies f^{(n)}(-2) = (-3)^n e^{6}$. This means that putting it all together, we have

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3)^n e^6}{n!}(x+2)^n. \tag{2}$$

(b) (5 points) **Compute the interval of convergence of your series in part (a). Name any tests or theorems that you used. You do *NOT* need to specify whether the seriese converges at the endpoints.**

**Solution:** Using the ratio test, we see

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{(-3)^{n+1} e^6 (x+2)^{n+1}}{(n+1)!} \times \frac{n!}{(-3)^n e^6 (x+2)^n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)(x+2)}{(n+1)} \right| \\ &= \lim_{n\to\infty} (-3)|x+2| \frac{1}{n+1} \\ &= (-3)|x+2| \lim_{n\to\infty} |\frac{1}{n+1} \\ &= 0 \end{aligned}$$

Since $0 < 1, \forall x \in \mathbb{R}$, the radius of convergence is $(-\infty, \infty)$.

(c) (5 points) **Suppose you wanted to use the Taylor polynomial $T_5(x)$ (still centered at -2) to approximate $e^4$. Find a bound on the error of this approximation. You do not need to simplify the arithmetic in your answer.**

**Solution:** Recall the bound for the error of a Taylor series is the first term that is ignored.

$$R_n(x) = \frac{M}{(n+1)!}(x-a)^{n+1} \tag{3}$$

So in our case $a = -2$ and $n = 5$. Also $|f^{(5+1)}(\xi)| \leq M$, where $\xi \in [a, x]$. We also see that $x = -4/3$ here since $f(x) = e^{-3(-4/3)} = e^4$, which is what we want to approximate. To find $M$ we need to see that $f^{(6)}(\xi) = (-3)^6 e^{-3\xi}$. To bound this we need to observe that the slope is the greatest when $\xi$ is smaller since $\xi \in [-2, -4/3]$. This means when $xi = 2$, the slope is going to be the biggest, so the error bound we get is

$$\begin{aligned} R_5(-4/3; -2) &\leq \frac{M}{(n+1)!}(x-a)^{n+1} \\ &= \frac{3^6 e^6}{6!}(-4/3+2)^6 \\ &= \frac{3^6 e^6}{6!}\left(\frac{2}{3}\right)^6 \end{aligned}$$