Phil's Algorithm (Comparing Quantum Representations)

- No Phil's algorithm (authors are being tongue in cheek)
- instead it's a schema of the quantum algorithms we will see going forward.

- "Given an X, the algorithm finds Y in time Z"
  - sometimes exactly, other times close enough with specified probability or expected length of time

§7.1 algorithm
- Schema: compute a series of vectors which to start off with are independent ~~of~~ of size of input X
  - This is because a vector relates to macrophase of algorithm which are small in number
- Will explain what Hilbert space we are in. mostly in real spaces, but QFT and Shor's algorithm uses complex Hilbert

§7.2 Analysis
- Analysis of algorithms are harder than descriptions, for quantum & classical algos
- Analysis offers a description of i^{th} vector
  ↳ unitary transforms to start vector
- Last step is quantum measurement. returns k with prob $|a_k|^2$ of last vector

- Some algos finish after measurement, or the measurement's value determines answer completely.
- Others take measurement and perform classic computations on top of it.

§ 7.3 Example

- Lets operate over 2D Hilbert space, $\mathbb{H}_2$

$\bar{a}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\bar{a}_1 = H_2 \bar{a}_0$ $H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\bar{a}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ $|a_0|^2 = 1/2$
$|a_1|^2 = 1/2$

§ 7.4 A Two Qubit example

$V_1 = H \otimes I$ $V_2 = CNOT$

Algorithm

1.) $\bar{a}_0$ so that $\bar{a}_0(00) = 1$ $a_0 = e_{00}$
2.) $\bar{a}_1$ is H_2 on qubit line 1 only
3.) $\bar{a}_2$ is $CNOT(\bar{a}_1)$

Analysis $\quad a_0 = [1\ 0\ 0\ 0]^T$

$$H_2 \otimes I_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}[1\ 0\ 1\ 0]$$

CNOT flips 3 & 4 coordinate

so $\quad \bar{a}_2 = \frac{1}{\sqrt{2}}[1, 0, 0, 1]$

$$= \frac{1}{\sqrt{2}}(e_{00} + e_{11})$$

$$a_1 = \frac{1}{\sqrt{2}}(e_0 + e_1) \otimes e_0 = \frac{1}{\sqrt{2}}(e_{00} + e_{10}) = \frac{1}{\sqrt{2}}[1, 0, 1, 0]$$

$$U = V_2 V_1 \Rightarrow U a_0 = V_2 V_1 a_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Hilbert ... just math
Quantumly
Is $\bar{a}_2 = \frac{1}{\sqrt{2}}(e_{00} + e_{11})$ tensor product of 2 states?

entangled?

when we finish algorithm we take measurement
- only get 00 or 11 not 01, 10
- if we measure first qubit and get 0 we know second qubit will give us 0 w/out measurement
- Alice & Bob measure far away violates physics.

Measurements

- Mouse example

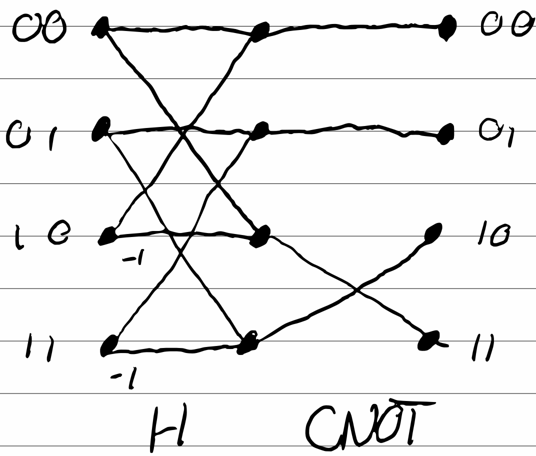
1.) superposition - when Phil goes to fork, 2 Phil
2.) interference - Phil + cheese = anti-phil
Phil + antiphil = 0
anti-phil + cheese = phil
3.) Amplification - 2 phils (anti-phils) meet and run along side each other
4.) measurement - at the end of the maze, put under incubator that grows it to (pack size)? then divid by # stages w/ cheese

that is the probability Phil excited there

5) state after measurement.

- after the measurement Phil becomes whole again.

Figure 7.1



probability: $\frac{(\text{packsize})^2}{2^h}$

$h = \#$ H gates

$\frac{1}{\sqrt{2}}$ 1001 Phil starts at 00 → 00 → 00, $\frac{1^2}{2}$, ↳ 10 → 11, $\frac{1^2}{2}$,

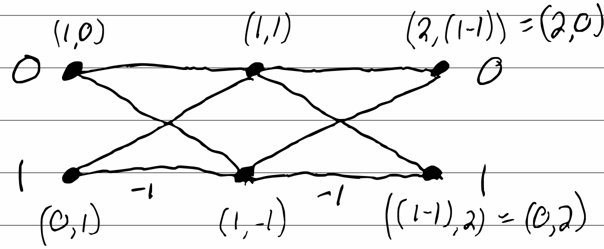
$\frac{1}{\sqrt{2}}$ 0110 Phil starts at 01 → 01 → 01, $\frac{1^2}{2}$, ↳ 11 → 10, $\frac{1^2}{2}$,

$\frac{1}{\sqrt{2}}$ 100-1 Phil starts at 10 → 00 → 00 $\frac{1^2}{2}$,

$\frac{1}{\sqrt{2}}$ 01-10 Phill starts at 11

Let's see some collisions!

Figure 7.2 two consecutive H gates



Out 0 | Phil starts at $0 \to 0 \to 0$ } pack of 2 so $\frac{2^2}{2^h} = 1.0$, $h = 2$
$\hookrightarrow 1 \to 0$

Out 1 | Phil starts at $0 \to 0 \to 1$ } Phil + anti phil $= 0$
$\hookrightarrow 1 \to -1$

Out 1 | Phil starts at $1 \to -1 \to 1$ } pack of 2 so $\frac{2^3}{2^2} = 1.0$
$\hookrightarrow 0 \to 1$

out 0 | Phil start at $1 \to -1 \to -0$ } $= 0$
$\hookrightarrow 0 \to 0$

in \ out	0	1
0	1.0	0
1	0	1.0

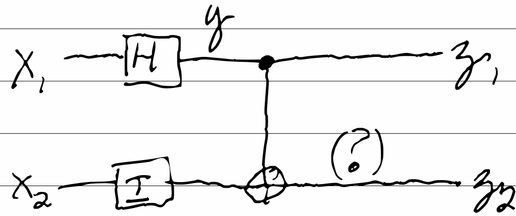
$$H \cdot H = H^2 = I$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

§7.6

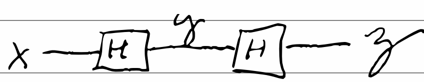
Quantum Mazes Vs Circuits Vs Matrices

- our maze diagrams scale with $N = 2^n$
- circuit diagrams scale with n instead



what is (?) as it could be 0 or 1 with equal probability (so $|1/\sqrt{2}|$ amplitude) Because of entanglement with y then can't say $\frac{e_0 + e_1}{\sqrt{2}}$

What about



doesn't show they cancel

Pros & Cons

Mazes: don't scale

Circuits: scale but make entanglement & info hard to trace

matrices: scale & preserves everything

mazes are exactly directed graphs corresponding to matrix products (§3.6) except adjacency matrices can have ± 1 entries

Mouse enters U in row i and leaves col j if $U[i,j] \neq 0$, if $U[i,j] = -1$ then it picks up cheese

§7.9 Summary & notes

- biggest problem rn is engineering problem of building QC that scale, which brings in physical noise that can muss up hair

- if components were small enough and went fast enough, then "noise" errors might be minimal or correctable.

Chapter 8: Deutsche Algo

X Y Z of Deutsches Algo

Given a FUNCTION, Deutsches Algo finds out if it is CONSTANT within 1 FUNCTION EVAL

- Why do we care?
  - 1^{st} nontrivial quantum algorithm
  - quantum algos can be more efficient than classical ones (minor but still)

Classical computation takes 2 fevals

§8.1 Algorithm

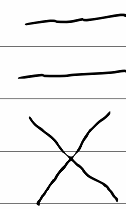
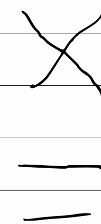
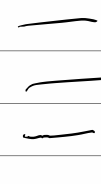
- computs a series of vectors $\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3$ where all are in real Hilbert space $\mathbb{H}_1 \times \mathbb{H}_2$ where $\mathbb{H}_1, \mathbb{H}_2$ are 2D space.

From 4.3 we recall $\forall$ Boolean f we can use invertible extension.

$$f'(xy) = x(f(x) \oplus y)$$

1.) input $\bar{a}_0$ is $a_0(01) = 1$
2.) $\bar{a}_1$ is result of multiplying H on $\mathbb{H}_i$ separately
3.) $\bar{a}_2$ is applying $U_{f'}$ where $f'(xy) = x(f(x) \oplus y)$
4.) $\bar{a}_3$ is applying H again on $\mathbb{H}_1$ only

Special case: f is identity f' is CNOT for $U_{f'}$ becomes 4x4 CNOT matrix

Function	U	Matrix	Maze
Identity	U_I	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	
X, negation	U_X	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	
Always true, T	U_T	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	
Always false, F	U_F	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	

NB U_T U_F are unitary but always-true -false aren't reversible.

This preserves "x" argument of these fxns as first qubit and f(x) when XOR fxn is applied to second qubit y

- We will sandwich U_I, U_X, U_T, U_F between

$$H_2 \otimes H_2 \qquad U_i \qquad H_2 \otimes I$$

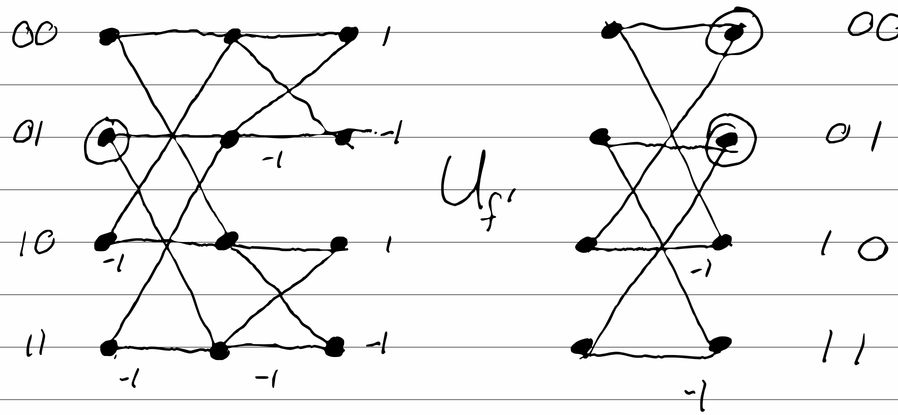
$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad U_i \quad = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

- Three matrices are applied to $\bar{a}_0 = \bar{e}_{01}$ on right
- Four cases for $\bar{a}_3$, and upon measurement we will see if we are in 2 constant cases U_T or U_F is used or whether we have U_I or U_X which are nonconstant functions, f.

Classically need to evaluate f(0) and f(1) but quantumly we just need one $U_{f'}$ oracle matrix where f' is "controlled" version of f.

§8.2 Analysis

Maze for Deutsch's algo



If we put U_f, constant false in
we see exciting at 00 is two phils
so positive amplitude 2 at 00.

we see exiting at 01 are two antiphils
so -2 in amplitud and 3 "cheese"
stages so

$00: 2^2/2^3 = 1/2$

$01: (-2)^2/2^3 = 1/2$

we see no room for 10, 11
we see both give Phil + antiphil = 0

Now lets look at U_T. U_T swaps both bottom and top so our analysis holds
now 00 has amplitude -2 and
01 has amplitude +2

both have probability 1/2 so only outcomes

U_I, U_X only do one swap.

This mean 00, 01 now have PAP cancelation so we get 1 in the first qubit!

S corresponding to 0 gives rise to 1-qubit measurement that perfectly distinguished constant function + non-constant function.

Thrm 8.1: Measurement of vecter $\bar{a}_3$ will return O_y for some y $\Leftrightarrow$ f is constant function. This implies Deutsch's algorithm tells wheather f is constant with one U_f.

Lemma 8.2: TFAT

1.) $\forall xy$, $a_1(xy) = \frac{1}{2}(-1)^y$

2.) $a_2(xy) = \frac{1}{2}(-1)^{f(x) \oplus y}$

3.) $|a_3(xy)|^2 = \frac{1}{8}\left|(-1)^{f(0)} + (-1)^{f(1) \oplus x}\right|^2$

Proof: Recall applying the Hadamard gate to a vector can be written as

$$\bar{b}(x) = \frac{1}{\sqrt{x}} \sum_{t=0}^{N-1} (-1)^{x \cdot t} a(t)$$

So applying them independently we get

$$\bar{a}_1(xy) = \frac{1}{2} \sum_{t,u} (-1)^{x \cdot t} (-1)^{y \cdot u} \bar{a}_0(tu)$$

Since $a_0 = e_{01}$ we get

$$a_1(xy) = \frac{1}{2}(-1)^{x \cdot 0}(-1)^{y \cdot 1} = \frac{1}{2}(-1)^y$$

Now $\bar{a}_2(xy) = a_1(x(f(x) \oplus y)) = \frac{1}{2}(-1)^{f(x) \oplus y}$

by definition of U_f.

Now $\bar{a}_3(xy) = \frac{1}{\sqrt{2}} \sum_t (-1)^{x \cdot t} \bar{a}_2(ty)$

$$= \frac{1}{2\sqrt{2}} \sum_t (-1)^{x \cdot t} (-1)^{f(t) \oplus y}$$

since $t = 0, 1$

$$\bar{a}_3(xy) = \frac{1}{2\sqrt{2}} \left((-1)^{f(0)+y} + (-1)^{x \oplus f(1) \oplus y} \right)$$

factoring out $(-1)^y$ we get

$$|\bar{a}_3(xy)|^2 = \frac{1}{8} \left| (-1)^{f(0)} + (-1)^{f(1) \oplus x} \right|^2$$

$\boxtimes$

Proof of Theorem 8.1
By lemma $|a_3(0y)|^2$ is

$$\frac{1}{8} \left| (-1)^{f(0)} + (-1)^{f(1)} \right|^2$$

if f is constant then this equals

$$\frac{1}{8} 2^2 = 1/2$$

else f is nonconstant then this equals

$$\frac{1}{8} 0^2 = 0$$

$\boxtimes$

§ 8.3 Superdense Coding & Teleportation

Relies only on H plus CNOT, not two Hadamard gates as in Deutsch's algo

Most basic forms of general construction called super dense coding and quantum teleportation
Relies on physical interpretation & relaxation of qubits

let c be the general product state st

$$c = (a_0\bar{e}_0 + a_1\bar{e}_1) \otimes (b_0\bar{e}_0 + b_1\bar{e}_1)$$
$$= a_0b_0\bar{e}_{00} + a_0b_1\bar{e}_{01} + a_1b_0\bar{e}_{10} + a_1b_1\bar{e}_{11}$$

where $|a_0|^2 + |a_1|^2 = |b_0|^2 + |b_1|^2 = 1$
Let Alice own $\bar{a} = a_0\bar{e}_0 + a_1\bar{e}_1$ and
let Bob own $\bar{b} = b_0\bar{e}_0 + b_1\bar{e}_1$ wholly

$$c = \bar{a} \otimes \bar{b}$$

general pure form state of system

$$\bar{d} = d_{00}\bar{e}_{00} + d_{01}\bar{e}_{01} + d_{10}\bar{e}_{10} + d_{11}\bar{e}_{11}$$

with $|d_{00}|^2 + |d_{01}|^2 + |d_{10}|^2 + |d_{11}|^2 = 1$

Three ways to partition these

Alice: controls first index so $d_{00}d_{01}$ v $d_{10}d_{11}$
Bob: controls second index so $d_{00}d_{10}$ vs $d_{01}d_{11}$

Another way: $d_{00}d_{11}$ VS $d_{01}d_{10}$

can be achieved directly by different kind of measurement that projects onto transformed basise whose elements are given by

$\frac{e_{00} \pm e_{11}}{\sqrt{2}}$, $\frac{e_{01} \pm e_{10}}{\sqrt{2}}$ named after John Bell

This means converting start vector $\bar{e}_{00}$ to

$$\bar{d} = \frac{1}{\sqrt{2}}\bar{e}_{00} + \frac{1}{\sqrt{2}}\bar{e}_{11}$$

we give alice a particle representing 1st cordinate & Bob across the lake an entangled particle representing the second coordinate.

Alice can operate further on this state by matrix operators applied only to her or Operators of the form $U \otimes I$

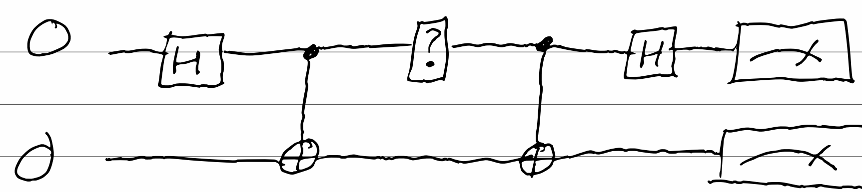
$U \in \mathbb{R}^{2\times 2}$

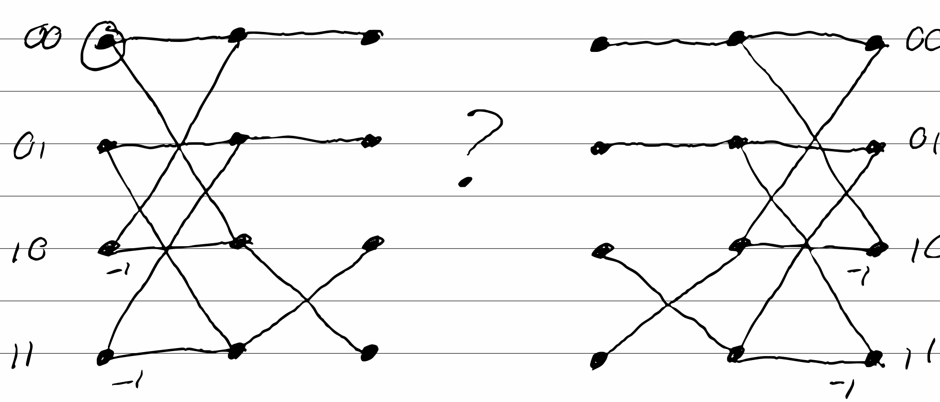
Now let U be i.) I ii.) X, iii.) Z or

(iv.) $XZ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -iY$

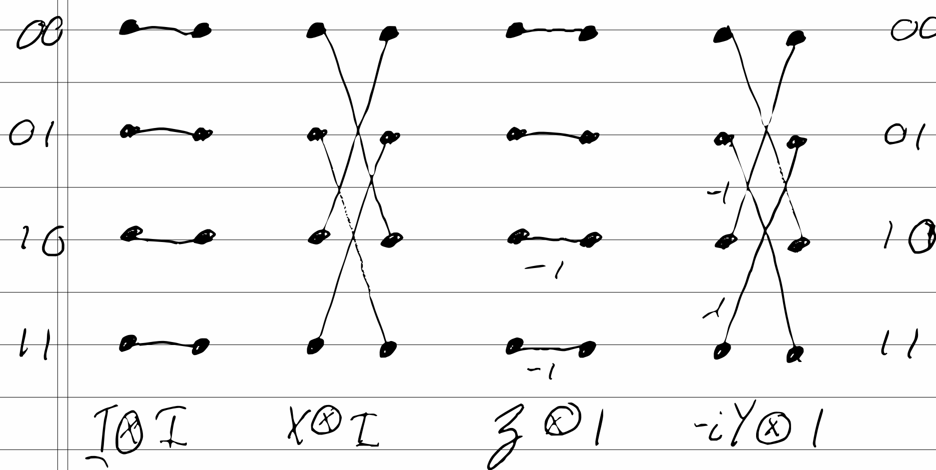
She performs $U \otimes I$ and ships it to Bob. Can Bob now able to carry out multi-qubit operations such as CNOT figure out what she did?

↳ He can "uncompute" the original entanglement and measure both qubits.





8.4 figure: Pauli operators on qubit 1



The exit point measurement depends only on what Alice chose. Alice's four choices lead to four different results, so Bob is able to tell what Alice did.

Bob learned 2 bits of information namely his and alices based only on Alices qubit. Did one qubit carry two pieces of classical information? No because there was a previous connection between them

Thrm: Holevo's Theorem: The total transmission of n qubits can carry no more than n bits of classical information.

There had to be prior interactions between them or their environments to produce entanglem... Once they are there, Alice can transmit information at a classically impossible 2 for 1 rate, but it does consume entanglement resources for each pair of bits.
This is "superdense coding"

Quantum teleportation involves 3 qubits Alices, Bobs entangled & Alice has another arbitrary (pure) state $\bar{c} = a_0\bar{e}_0 + b\bar{e}_1$ Alice has no knowledge of this state.