Haar measure and Integration over Lie Groups

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1 Introduction

When taking a random matrix theory course, there were many generalizations to standard mathematical ideas that had to be formulated for these random, noncommuting objects. One of which was the HCIZ integral, which is a generalization of the Fourier Transform for matrices.

Definition 1.1 (Harish-Chandra-Itzykson-Zuber (HCIZ) integral)**.** The Harish-Chandra-Itzykson-Zuber (HCIZ) integral is:

$$I_\beta(\mathbf{A},\mathbf{B}) := \int_{G(N)} \mathrm{d}\mathbf{U}\, e^{\frac{N\beta}{2}\operatorname{Tr}\mathbf{AUBU}^\dagger}, \qquad \int_{G(N)} \mathrm{d}\mathbf{U} = 1 \tag{1}$$

where the integral is over the Haar measure of a compact group $\mathbf{U} \in G(N) = O(N), U(N)$, or $\mathrm{Sp}(N)$ in N dimensions and $\mathbf{A}, \mathbf{B}$ are arbitrary $N \times N$ symmetric, Hermitian, or symplectic matrices with $\beta = 1, 2$, or 4, respectively.

Remark. $I_\beta(\mathbf{A},\mathbf{B})$ depends only on the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, since if we change the basis on either $\mathbf{A}$ or $\mathbf{B}$, it can be absorbed into $\mathbf{U}$, and we are integrating over them.

While it might not look like we can do much generalizing on this HCIZ integral, the celebrated result in [11, 12] is that for $G(N) = U(N)$ which implies that $\beta = 2$, the integral can be exactly expressed for all $N \in \mathbb{N}$. In fact,

$$I_2(\mathbf{A},\mathbf{B}) = \frac{c_N}{N^{(N^2-N)/2}} \frac{\det\left(e^{N\nu_i\lambda_j}\right)}{\Delta(\mathbf{A})\Delta(\mathbf{B})},$$

where $\{\nu_i\}, \{\lambda_i\}$ are the eigenvalues, and $\Delta(\mathbf{A}), \Delta(\mathbf{B})$ are the Vandermonde determinant of $\mathbf{A}, \mathbf{B}$, respectively. Lastly, $c_N = \prod_\ell^{N-1} \ell!$.

But what is this mysterious $\mathrm{d}\mathbf{U}$? The book we used in my random matrix theory class [15] referenced [4], when first discussing the integration of groups of physical interest. However, it was far too advanced, so to better understand it, the Haar Measure will be considered in great detail. For an alternative way to motivate the HCIZ integral and Haar Measure, the interested reader can check out [2]. This is just a note set that I prepared to better understand Haar measure and integration over matrix groups, and I am by no means an expert. I hope this can be a gentle introduction into the topic for new readers. Also, if there are any typos, feel free to reach out to me so that I can correct them.

2 Haar Measure

2.1 History

Introduced by Alfred Haar in 1933, but for Lie groups it was introduced by Adolf Hurwitz in 1897, under the name "invariant integral". Existance and uniquness was proven by Andre Weil in 1938 using AoC, but Henri Cartan does it without AoC in 1940.[1].

2.2 Formal Setting of Haar Measure

Definition 2.1 (Borel measure)**.** A measure μ on a topological measure space X is called a *Borel measure* iff X is Hausdorff.

Definition 2.2 (Left Haar Measure)**.** Let G be a topological group. A *left Haar Measure* on G is a nonzero regular Borel measure μ on G such that $\mu(gA) = \mu(A), \forall g \in G$ and for all measurable subsets A of G.

Definition 2.3 (Right Haar Measure)**.** There is also a *right Haar measure* with the same conditions as above just with $\mu(Ag) = \mu(A), \forall g \in G$.

Remark (Existence and Uniqueness of Haar Measure)*.* While existence and uniqueness are vital properties, their proof is extremely nontrivial, as you have to go through the typical process of verifying dμ is a measure by premeasure to outer measure to measure. It is a long but rewarding process. However, it is not the point of my introduction to Haar measure, so it will be assumed. If you would like to learn more, [8, 9] gives a very detailed proof of both properties from a measure-theoretic and topological perspective. Additionally, [3] proceeds from a combinatoric angle to prove the uniqueness using hypergraphs.

Theorem 2.1 (Existence, (Weil))**.** *Let G be a locally compact (Haussdorff topological) group. Then there exists a left Haar measure on G.*

Corollary 2.2. *Since μ is a measure, it has the following properties:*

1. *the measure μ is finite on every compact set: $\mu(K) < \infty$ for all compact $K \subseteq G$.*
2. *the measure is regular, meaning for any Borel set S,*

$$\mu(S) = \inf\{\mu(U) : S \subseteq U \textit{ open }\} = \sup\{\mu(K) : \textit{compact } K \subseteq U\}$$

3. *For every non-empty open subset $U \subseteq G$, then $\mu(U) > 0$.*

Remark. On an n-dimensional Lie group, Haar measure can be constructed easily as the measure induced by a left-invariant n-form. This was known before Haar's theorem.

Theorem 2.3 (Uniqueness, (Weil))**.** *Let G be a locally compact group, and let μ and μ' be two left Haar measures on G. Then, $\mu = \alpha\mu'$ for some $\alpha \in \mathbb{R}_{>0}$.*

While the left and right Haar measure don't have to be the same, we can find a relationship between them:

Corollary 2.4. *Let G be a topological group. Also, let μ_L be a left Haar measure. For a Borel set S, we define S^{-1} as the set of inverse of elements of S. Then $\mu_R(S) = \mu_L(S^{-1})$ is a right Haar measure.*

Proof. Let G be a topological group. For some Borel $S \subset G$, and $g \in G$

$$\mu_R(Sg) = \mu_L((Sg)^{-1}) = \mu_L(g^{-1}S^{-1}) = \mu_L(S^{-1}) = \mu_R(S).$$

Additionally, we should show that μ_R is a regular Borel Measure. □

Another easily seen property is that the left invariant Haar measures also behave well under right multiplication [19].

Example 2.5. *Let μ_L be a left Haar measure on G. Then for any $x \in G$, we define μ'_L on Borel subsets $S \subseteq G$ by $\mu'_L(S) = \mu_L(Sx)$. This μ'_L is still a left Haar Measure. Now for any $y \in G$,*

$$\mu'_L(yS) = \mu_L(ySx) = \mu_L(Sx) = \mu'_L(S).$$

This means by the uniqueness of the Haar measure up to a multiplicative constant, we see there is a positive number $\Delta(x)$ such that $\mu'_L = \Delta(x)\mu_L$.

Similarly right invariant Haar measures behave well under left multiplication.

Example 2.6. *Let μ_R be a right Haar measure, then for a fixed choice of a group element $x \in G$, and a Borel subset $S \subseteq G$., $S \mapsto \mu_R(x^{-1}S)$ is a right invariant Haar meausre. Again by uniqueness, we see $\mu_R(x^{-1}S) = \Delta(x)\mu_R(S)$, and this $\delta(x)$ is independent of the Haar measure.*

Definition 2.4 (Modular function)**.** The *modular function* $\Delta : G \to \mathbb{R}^+$ is a continuous Lie group homomorphism that takes values $\mu_L(Sx) = \Delta(x)\mu_L(S), \forall S \subset G$.

However sometimes the left and the right Haar measure agree and are the same, which motivates:

Definition 2.5 (Unimodular)**.** Let G be a locally compact group. If the left and right Haar measure agree, we say that G is *unimodular*.

Take either of the above examples, such as $\mu'_L = \Delta(x)\mu_L$. μ'_L is still a left Haar measure but it behaves nicely with right multiplication. So if $\Delta(x) \equiv 1$ for all $x \in G$, we see

$$\mu_L(Sx) = \mu'_L(S) = \Delta(x)\mu_L(S) = \mu(S),$$

so μ_L is right invariant.

Corollary 2.7. *A Lie group G is unimodular iff $|\det Adg| = 1$, for all $g \in G$,where $Ad : G \to End((g))$ is the adjoint representation of G. Also if G is connected, we can remove the absolute value bars [6].*

Remark. When the group is unimodular, we see something super cool.

$$\int_G f(hg)\,\mathrm{d}\mu(g) = \int_G f(gh)\,\mathrm{d}\mu(g) = \int_G f(g^{-1})\,\mathrm{d}\mu(g) = \int_G f(g)\,\mathrm{d}\mu(g)$$

Or the left invariant integral is the right invariant integral is the inversion integral is the integral.

Example 2.8 (Linear Affine Transformation)**.** *Recall that the linear affine transformation is given by*

$$\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}_{>0}, b \in \mathbb{R}\right\}$$

To define the left Haar measure, we see that

$$\begin{aligned}
\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} ac & ad+d \\ 0 & 1 \end{pmatrix} \\
\begin{pmatrix} c & d \end{pmatrix} &\mapsto \begin{pmatrix} ac & ad+b \end{pmatrix} \\
J_L &= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \\
\det(J_L) &= a^2 \\
\mu_L &= a^{-2}\,\mathrm{d}b\,\mathrm{d}a
\end{aligned}$$

Now computing the right Haar measure

$$\begin{aligned}
\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} ac & bc+d \\ 0 & 1 \end{pmatrix} \\
\begin{pmatrix} c & d \end{pmatrix} &\mapsto \begin{pmatrix} ac & bc+d \end{pmatrix} \\
J_R &= \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \\
\det(J_R) &= a \\
\mu_R &= a^{-1}\,\mathrm{d}b\,\mathrm{d}a
\end{aligned}$$

Remark. We don't want the measure to be negative so if we slightly change the example of the affine linear transformation to $a \in \mathbb{R}\backslash\{0\}$, then the left Haar measure is the same but the right Haar measure is $\mu_R = |a^{-1}|\,\mathrm{d}b\,\mathrm{d}a$ so that we don't have a orientation reversing measure.

Example 2.9. *Now let's look at $GL_n(\mathbb{R})$. Recall that for $GL_2(\mathbb{R})$ is*

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \textit{matrix is invertible}\right\}$$

The left invariant measure preserves volumes, and we know that $|\det(G)|$ is the change in volume of the vector, so we know

$$\mu_L = \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}d}{\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|}$$

Now if we consider $g \in GL_n(\mathbb{R})$, it is not hard to see

$$\mu_L = \frac{\mathrm{d}g}{|\det(g)|^n},$$

where $\mathrm{d}g$ is the Lebesgue measure in $\mathbb{R}^{n^2}$. This follows from the change-of-variable formula. Exercise to show this is also the right Haar measure.

Remark. These are very standard examples of a non-unimodular group being compared to a uimodular group. Most texts on Haar measure use these two.

Now that we have seen some examples of unimodular groups, and exactly what groups are unimodular, let's solidify this concept.

Corollary 2.10. *A Lie group is unimodular if it is discrete, Abelian, or compact.*

Remark. As opposed to the last result on unimodular groups, this is simply an "if" proof. Clearly the "only if" direction is violated as we have already shown that $GL_n(\mathbb{R})$, which is a noncompact group, is unimodular.

Proof. For a discrete group, both the left and right Haar measure are the counting measure. For the abelian group, left- and right-translation actions are the same. Now for a compact group G we see that μ_L is finite for all compact sets and $Gx = G$ so

$$\mu_L(G) = \mu_L(Gx) = \Delta(x)\mu_L(G)$$

and dividing through by a finite number we see $1 = \Delta(x), \forall x \in G$. □

Remark. To be concrete, the following Lie groups are unimodular:

- All Finite Groups
- Abelian groups like $\mathbb{R}, \mathbb{R}^\times$
- Orthogonal group (compact)
- Unitary group (compact)
- classical semisimple Lie groups like $\mathrm{Sp}_n(\mathbb{R})$.

You can also show that other Lie groups that are unimodular are those that are connected and semisimple, connected and reductive, and connected and nilpotent. While there are more Lie groups that are unimodular, we have now shown that all $O(N), U(N), \mathrm{Sp}(N)$ are unimodular, which was an initial goal in understanding the HCIZ integral.

3 Examples

Theorem 3.1. *Let G be a compact topological group with Haar measure μ. If G is finite, then $\mu(\{g\}) = \frac{1}{|G|}$ for any $g \in G$. If G is infinite, then $\mu(\{g\}) = 0$ for any $g \in G$.*

Proof. Let $\mathcal{G}$ be some finite topological group. Then the group is compact as it is finite. Since it is compact, we can uniquely identify the Haar measure by normalizing it, so this means there is a probability measure on $\mathcal{G}$. Since G is Hausdorff, we assumed it was T_1, and the singleton points are closed. So for all $g, h \in \mathcal{G}$,

$$\mu(\{g\}) = \mu(hg^{-1}\{g\}) = \mu(\{h\})$$

by the invariance of the Haar measure. This means that $(G) = \cup_{g\in\mathcal{G}}\{g\}$, which are all disjoint from each other and singleton sets. Recalling that the Haar measure is also countably additive, we observe,

$$\mu(G) = \mu\left(\bigcup_{g\in\mathcal{G}}\{g\}\right) = \sum_{g\in\mathcal{G}}\mu(\{g\}) = 1$$

This implies that $\mu(\{g\}) = \frac{1}{|G|}$. Then as a simple corollary, we have the measure of $A \subset G$ as $\mu(A) = \frac{|A|}{|G|}$, as the sum of the measures of the points in A.
Reiterating, this shows that the unique Haar measure on a finite compact group is the uniform probability measure, as it is compact, so we can uniquely normalize it so $\mathrm{d}\mu = 1$, and it is unimodular so $\mu_L = \mu_R$.
Now let G be an infinite compact group. Suppose that μ is the Haar measure on G. Then by the invariance, we see $\mu(\{g\}) = \mu(\{h \cdot \{g\}\})$ for any g. This give us

$$\mu(\{g_1\}) = \mu(h \cdot \{g_1\}) = \mu(g_2 g_1^{-1} \cdot \{g_1\}) = \mu(\{g_2\})$$

where $h = g_2 g_1^{-1}, \forall g_1, g_2 \in G$. Now if $\mu(\{g\}) > 0$ for some g, then for some countable sequences of $g_i \in G$ we see $\sum_{i=1}^{\infty} \mu(\{g_i\}) = \mu(G) = 1$ because every g_i is disjoint. But we said that $\mu(\{g\}) = c > 0$, and any constant nonzero sequence sums to ∞. This means $\mu(g) = 0$. □

Theorem 3.2. *If G is a topological group, μ is the Haar measure and H is a $\mu-$measurable subgroup, the $\mu(H) = \frac{1}{|G/H|}$ if $|G/H| < \infty$ or $\mu(H) = 0$ if $|G/H| = \infty$.*

Proof. Let H be a subgroup of the compact topological group G, with Haar measure μ. Then we know $G = \bigcup_{g \in G} g \cdot H$. Inclusion in one of these co-sets form an equivalence relation on G, where $a \sim b \iff b \in a \cdot H$.
Then each element of G/H is a left translation of H, for any $g \cdot H, g' \cdot H \in G/H$, we have $\mu(g \cdot H) = \mu(g' \cdot H)$, and these elements are disjoint, from the last theorem, we see if G/H is infinite then $\mu(H) = 0$ or G/H is finite and $\mu(H) = \frac{1}{|G/H|}$. □

Both of these are proven similarly in [17].

Remark. To explicitly compute the Haar Measure:

- First, check if it is unimodular.
- Find a coordinate system for G and translate the group law into this coordinate system.
- Do multivariable calculus.

Example 3.3. *Consider $U(1) = \{e^{i\theta} : 0 \leqslant \theta < 2\pi\}$. Then to integrate*

$$\int_{U(1)} f(\gamma)\,\mathrm{d}\mu(\gamma) = \int_0^{2\pi} f(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi}$$

Example 3.4. *Consider $SU(2)$ which is equivalent to S^3 as a topological space.*

$$g = \begin{pmatrix} x_1 + ix_4 & x_2 + ix_3 \\ -x_2 + ix_3 & x_1 - ix_4 \end{pmatrix},$$

where $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. Then using the spherical coordinates in $\mathbb{R}^4$, we have

$$\begin{aligned} x_1 &= r\cos\theta \\ x_2 &= r\sin\theta\cos\psi \\ x_3 &= r\sin\theta\sin\psi\cos\phi \\ x_4 &= r\sin\theta\sin\psi\sin\phi, \end{aligned}$$

then we see

$$\left|\det \frac{\partial(x_1, x_2, x_3, x_4)}{\partial(r, \theta, \psi, \phi)}\right| = r^3 \sin^2\theta \sin\psi,$$

Recall $r^2 = 1$. *Then when we finish normalizing we see*

$$\int_{\theta=0}^{\pi}\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi} f(\theta,\psi,\phi)\,\mathrm{d}\mu\,, \text{ where } \mathrm{d}\mu = \frac{\sin^2\theta\sin\psi\,\mathrm{d}\theta\,\mathrm{d}\psi\,\mathrm{d}\phi}{2\pi^2}$$

Example 3.5. *Lastly, we finish with* $SO(3)$, *which we can think of as rotations in* $3-d$ *space, so an arbitrary element* $R \in SO(3)$ *can be factored into:*

$$R = ABC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{pmatrix} \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where θ, ϕ, ψ *are called "Euler angles" and are the rotation around the* $x-, y-,$ *and* $z-$*axis respectively. The angles* $\theta, \phi \in [0, 2\pi), \psi \in [0, \pi)$. *By performing the determinant of the Jacobian, we see that the differential becomes*

$$|\det J| = \sin(\psi).$$

Finally, we normalize to see

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi}\int_{\psi=0}^{\pi} f(\theta,\phi,\psi)\,\mathrm{d}\mu\,, \text{ where } \mathrm{d}\mu = \frac{\sin(\psi)\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\psi}{8\pi^2}$$

Remark. These computations are very popular ones as it has many physical interpretations, and was commputed separately and slightly differently in [13, 14, 20].

While I could continue to do computations of the (invariant) Haar measure for $U(N), SU(N), O(N), SO($
etc., there is actually a generalized formula that was found.

Theorem 3.6 (Haar Measure of Unitary Group, [18])**.** *The infinitesimal volume element* $\mathrm{d}\Omega$ *of Haar measure of unitary group of* n*-th order is given by the following formula.*

$$\mathrm{d}\Omega = \left|I_n + H^2\right|^{-n} \mathrm{d}h\,,$$

where H *is the Cayley parameter of unitary matrix*

$$U = (I_n + iH)(I_n - iH)^{-1}$$

and Hermitian, so that

$$\tilde{H} = H = (h_{ij}) = (a_{ik+ib_{ik}}), (a_{ik} = a_{ki}, b_{ki} - b_{ik})$$

and $\mathrm{d}h$ *is the product of all differentials of* n *parameters,*

$$\mathrm{d}h = \mathrm{d}a_{11}\,\mathrm{d}a_{12}\cdots\mathrm{d}a_{nn}\,\mathrm{d}b_{21}\cdots\mathrm{d}b_{n,n-1}\,.$$

Theorem 3.7 (Haar Measure of Symplectic Group, [18])**.** *For unitary symplectic group of* $2n$*-th order, the Haar Measure possesses the following form:*

$$\mathrm{d}\Omega = \left|I + H^2\right|^{-\left(n+\frac{1}{2}\right)} \mathrm{d}h\,.$$

Theorem 3.8 (Haar Measure of Orthogonal Group, [18])**.** *The Haar Measure of orthogonal group of* m*-th order is expressed as follows:*

$$\mathrm{d}\Omega = \left|I + H^2\right|^{-\frac{n-1}{2}} \mathrm{d}h\,.$$

4 Bibliography

As mentioned, I come from an applied math perspective and have taken very few algebra classes, so most of this had to be learned to understand Haar Measure. While I cited the resources consulted throughout this expository piece, there were other articles and books that I used to get a solid foundation for the material. These were used much less than the sources cited above, but I consulted the standard measure theory books to refresh my memory on certain topics. These include Stein and Shakarchi, Folland, and Rudin. Some more specialized resources I used sparingly are Conway's functional text[5], Easton's book on group invarance in statistics has a chapter on Haar Measure [7], and Hall's book on Lie Groups [10]. Additionally, there is a fantastic video series on Lie Groups on Youtube. This specific video [16] is very quick and was a great motivation for my presentation of this material. Lastly, the structure of this seems to follow a standard treatment of Haar Measure as set forth by [6].
This is not at all comprehensive. There are so many more ways and directions that this can go. For example, if I hade more time I would like to look into Kakutani's Fixed Point Theorem, Iwasawa decompositions, Peter-Weil Theorem, and the Schur Lemma.

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