On the Spectrum of Beta and Dirichlet Random Matrices

With Applications to Compressed Sensing

Mitchell Scott

Department of Mathematics, Emory University

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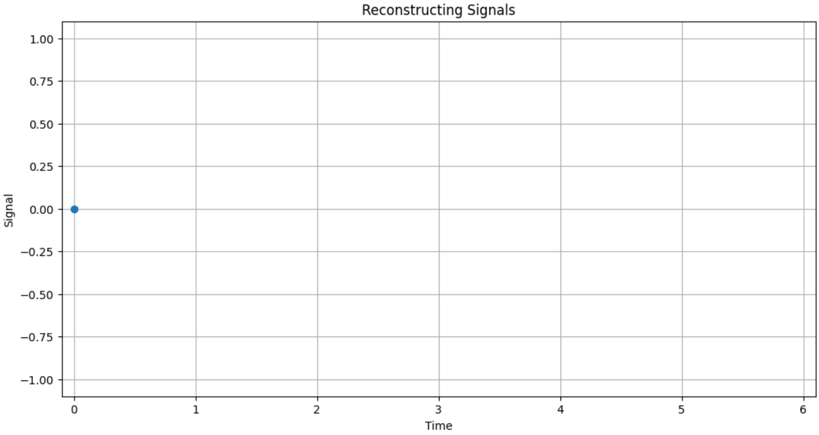
Sampling Problem



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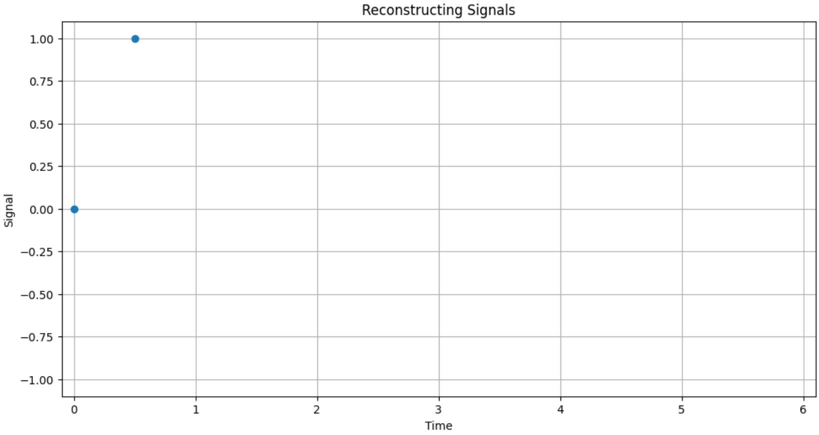

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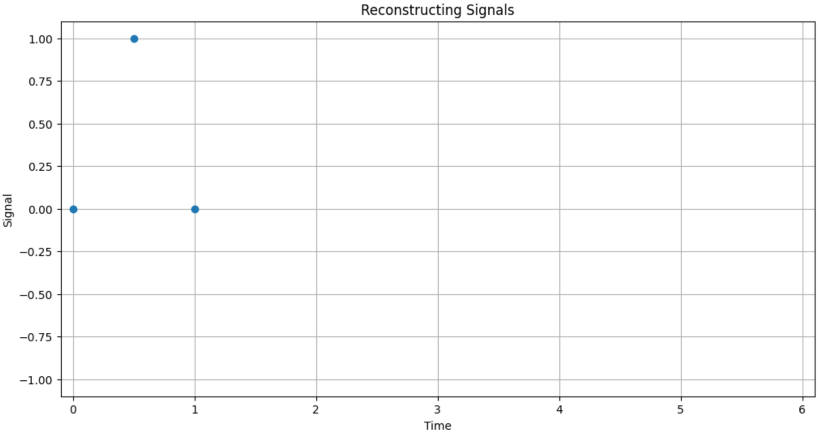

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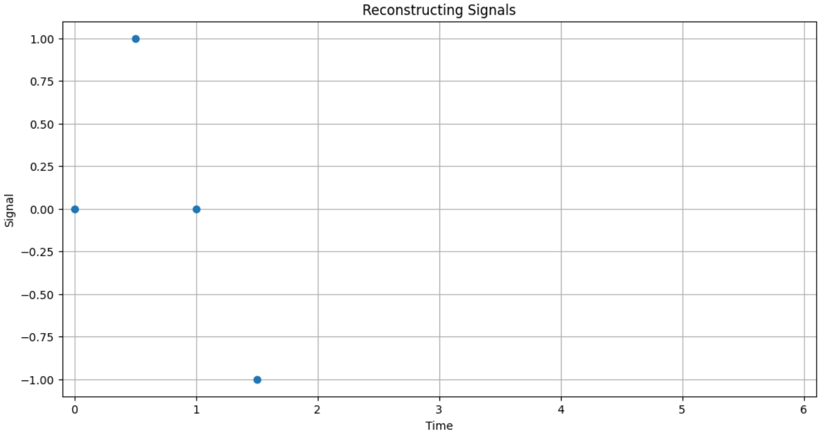

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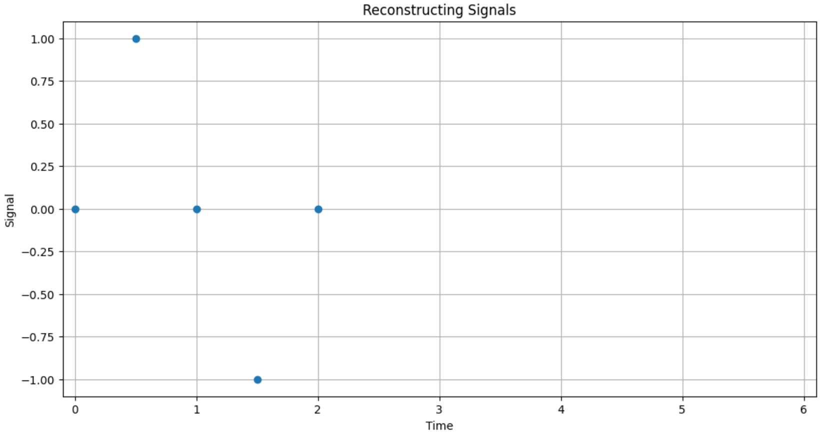

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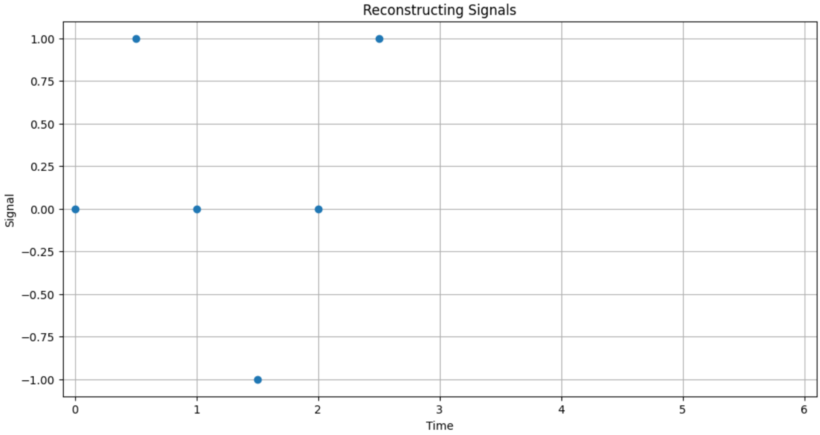

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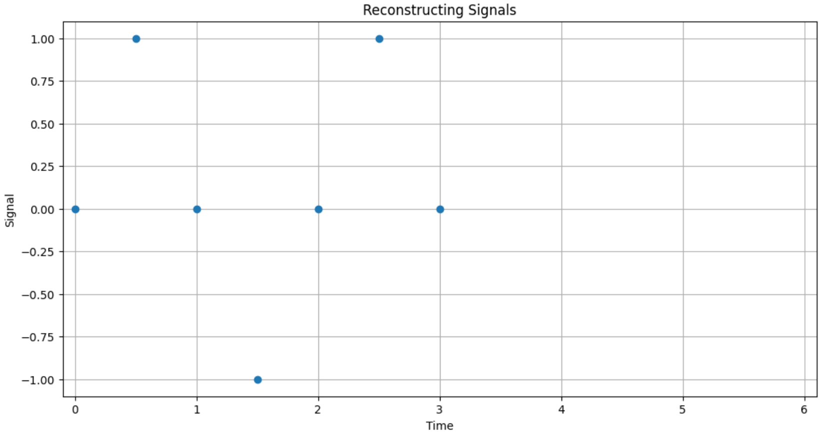

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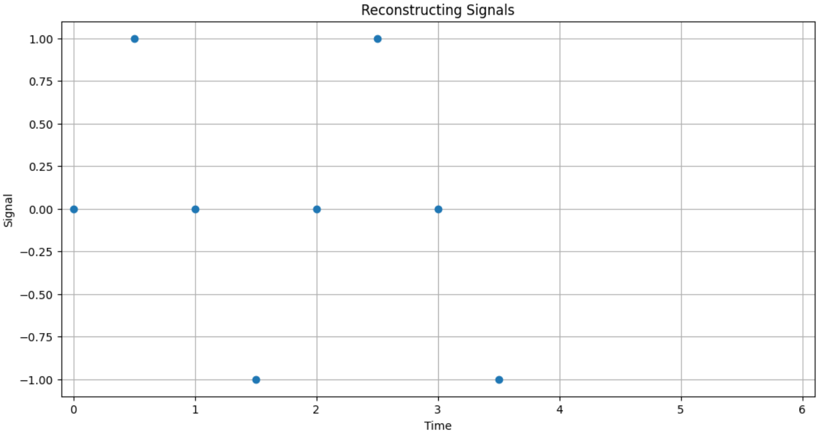

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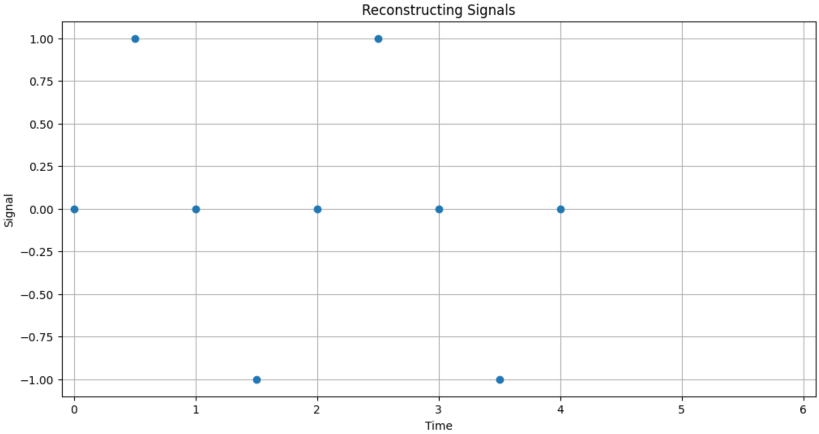

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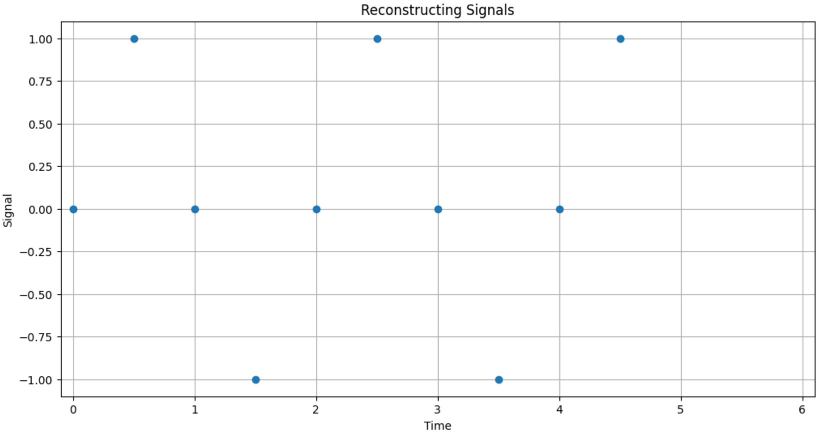

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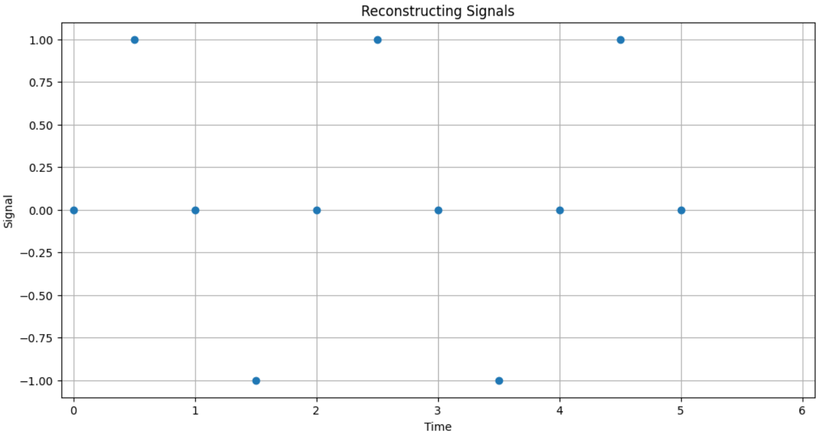

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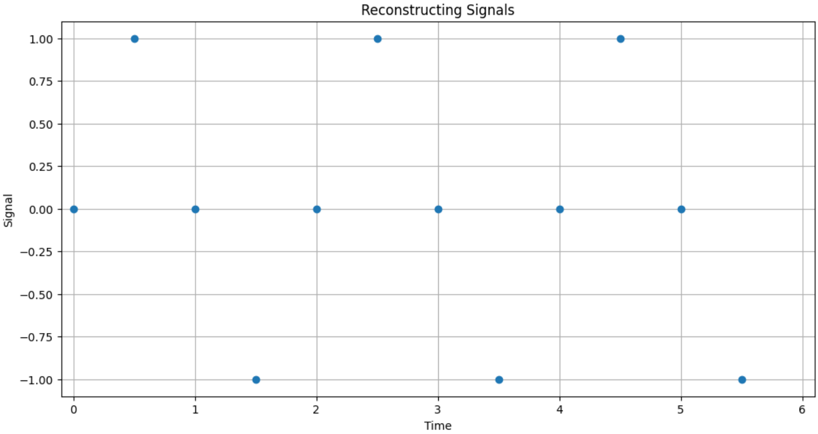

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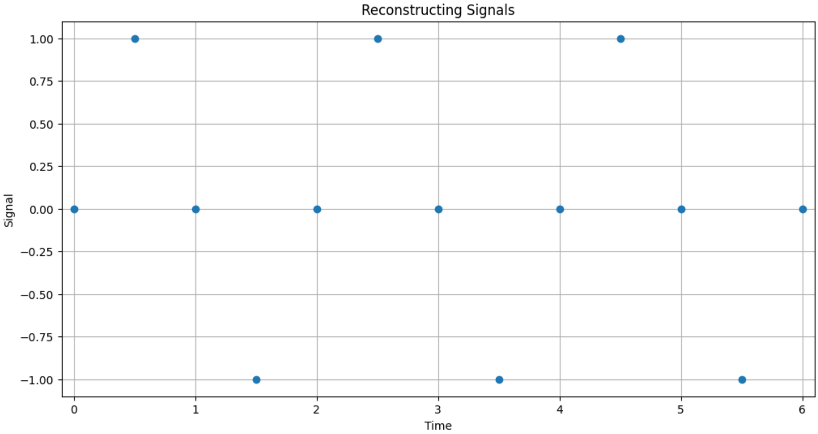

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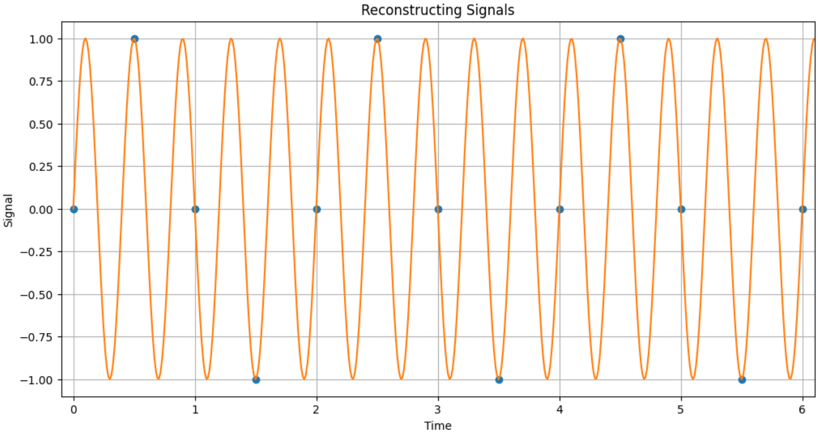

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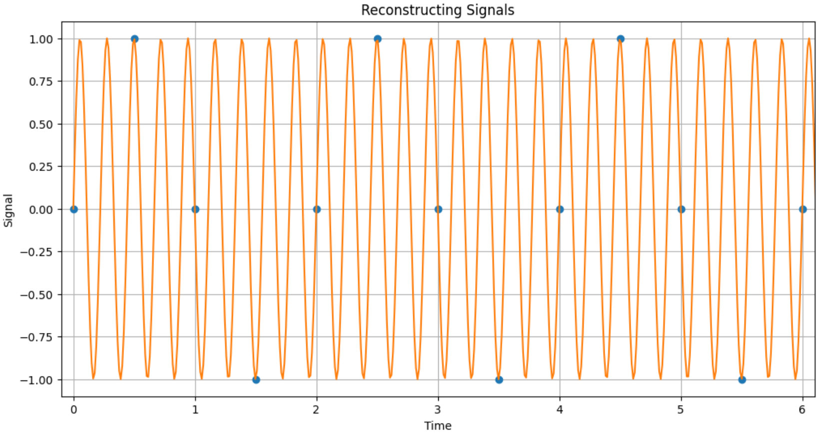

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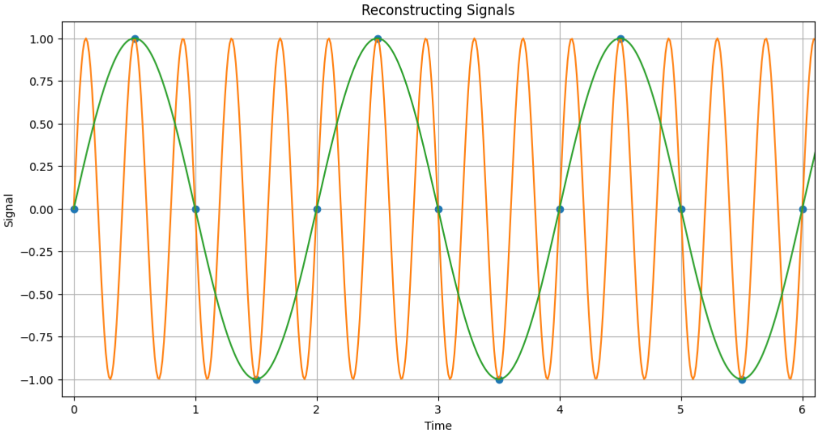

Solution

Theorem (Nyquist - Shannon Sampling Theorem, 1915, [1])

If a function $x(t)$ contains no frequences higher than B hertz $[s^{-1}]$, then it can be completely determined from sampling a sequence of points spaced less than $1/(2B)$ seconds apart.

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Key Take-away

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Key Take-away

In systems where you want to generate accurate signals from sampling data, you must set the sampling rate high enough to prevent aliasing.

Remark

However, the number of samples needed for high frequency data or long range signals, might be memory intesive to store explicitly. Additionally, how can you set up the sampling rate a priori?

Compressed Sensing

Remark

Perfect reconstruction of a signal can happen even if the N-S criterion isn't satisfied!

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Definition (Compressed Sensing)

Compressed sensing is a signal processing technique for efficiently acquiring and reconstructing a signal, by finding solutions to underdetermined linear systems.

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Compressed sensing is a signal processing technique for efficiently acquiring and reconstructing a signal, by finding solutions to underdetermined linear systems.

Theorem (Candes, Romberg, Tao (2005)[2])

Given some knowledge about a signal's sparsity, the signal may be reconstructed with even fewer samples than the sampling theorem requires, the basis of compressed sensing.

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Wishart Matrices

Definition (Wishart Ensemble)

The data for a *Wishart Ensemble* is a matrix of $N \times T$ data $\{x_i^t\}_{1 \leq i \leq N, 1 \leq t \leq T}$, where we have T observations and each observation contains N variables.

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Wishart matrices can arise in many examples, such as:

- daily returns of N stocks over a certain time period,
- number of spikes fired by N neurons during T consecutive intervals of Δt,
- and so many more.

Sample Covariance Matrices

Definition (Sample Covariance Matrix)

The *sample covariances* of the data are given by

$$\mathbf{E}_{ij} = \frac{1}{T}\sum_{t=1}^{T} x_i^t x_j^t. \tag{1}$$

This results in an $N \times N$ matrix $\mathbf{E}$, called the *sample covariance matrix*, which can be written as

$$\mathbf{E} = \frac{1}{T}\mathbf{H}\mathbf{H}^\top, \tag{2}$$

where $\mathbf{H}$ is the $N \times T$ matrix with $\mathbf{H}_{it} = x_i^t$.

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Convergence of Wishart Matrices' Spectrum

Theorem (Marčenko-Pastur [3])

The full Marčenko-Pastur distribibution can be written as such, let $\mathbf{M} \in \mathbb{R}^{N \times T}$, *where* $N, T \to \infty, N/T \to q \in (0, \infty)$. *Now let* $\lambda_{\pm} = \sigma^2 \left(1 \pm \sqrt{q}\right)^2$. *Then the density of the eigenvalues of* $\mathbf{M}$ *converges weakly to*

$$\rho_{MP}(x) = \frac{1}{2\pi\sigma^2} \frac{\sqrt{\left[(\lambda_+ - x)(x - \lambda_-)\right]_+}}{2\pi q x} \tag{3}$$

where $[a]_+ := \max\{a, 0\}$.

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$$\rho_{MP}(x) = \frac{1}{2\pi\sigma^2} \frac{\sqrt{\left[(\lambda_+ - x)(x - \lambda_-)\right]_+}}{2\pi q x} + \left[\frac{q-1}{q}\right]_+ \delta(x) \tag{3}$$

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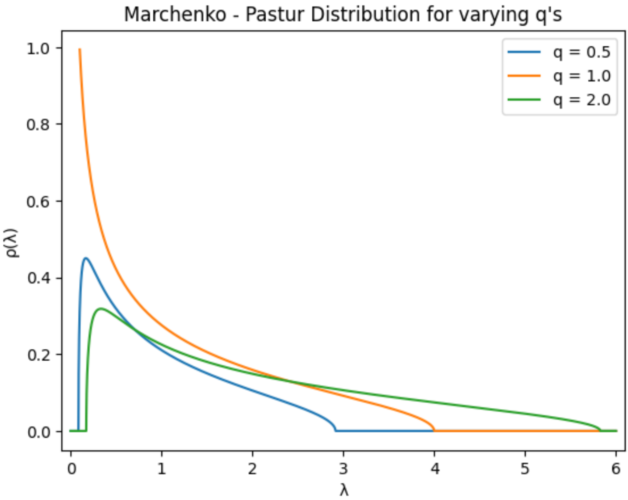
Visualizing the Marčenko-Pastur Distribution



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Beta Distribution

Definition (Beta Distribution PDF)

Let $x \in [0, 1]$. The Beta distribution has two shape parameters $\boldsymbol{\beta} = [\beta_1, \beta_2]^\top$, which control the growth of small and large values of x, respectively. Then the *Beta Distrbution PDF* is given by

$$f(x; \boldsymbol{\beta}) = \frac{1}{B(\boldsymbol{\beta})} x^{\beta_1 - 1} (1 - x)^{\beta_2 - 1}, \quad \text{where} \tag{4}$$

$$B(\boldsymbol{\beta}) = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1 + \beta_2)} \tag{5}$$

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Example

To sample a vector $\mathbf{v} \in \mathbb{R}^t$ from this distribution, we draw samples from the Beta distribution with parameters $\boldsymbol{\beta}$, and then rescale all samples by $\sum_{i=1}^t \mathbf{v}_i$. This ensures that $\|\mathbf{v}\|_1 = 1$.

Dirichlet Distribution

Definition (Dirichlet Distribution PDF)

Also known as the multivariate Beta distribution, the *Dirichlet Distribution PDF* of order $K \geq 2$ with parameters $\boldsymbol{\beta} = [\beta_1, \cdots, \beta_K]^\top$ is given by

$$f(\mathbf{x}; \boldsymbol{\beta}) = \frac{1}{B(\boldsymbol{\beta})} \prod_{i=1}^{K} x_i^{\beta_i - 1}, \quad \text{where} \tag{6}$$

$$B(\boldsymbol{\beta}) = \frac{\prod_{i=1}^{K} \Gamma(\beta_i)}{\Gamma\left(\sum_{i=1}^{K} \beta_i\right)}, \tag{7}$$

and $x_i \in [0, 1], \forall i \in \{1, 2, \cdots, K\}$, subject to $\sum_{i=1}^{K} x_i = 1$.

Sampling from Dirichlet Distribution

Example

While sampling from the Beta distribution is available in `numpy`, Dirichlet distributions are not. However, we do have access to the Gamma distribution. This means to sample a random vector $\mathbf{x} = [x_1, x_2, \ldots, x_K]^\top$ from the K-dimensional Dirichlet distribution with parameters $\boldsymbol{\beta} = [\beta_1, \beta_2, \ldots, \beta_K]^\top$, we draw K independent samples from Gamma distribution and normalize to sum to 1.

$$\text{Gamma}(\beta_i, 1) = \frac{y_i^{\beta_i - 1} e^{-y_i}}{\Gamma(\beta_i)} \tag{8}$$

$$x_i = \frac{y_i}{\sum_{j=1}^{K} y_j} \tag{9}$$

Visualizing these Vectors

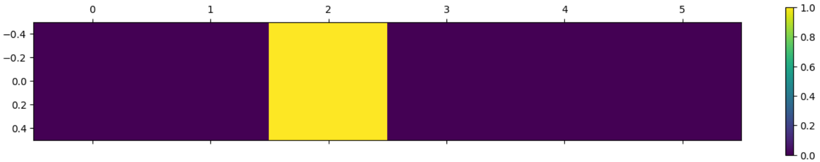

Figure 2: Small values of $\boldsymbol{\beta} = [0.0001, 0.0001]^\top$ promote sparcity.

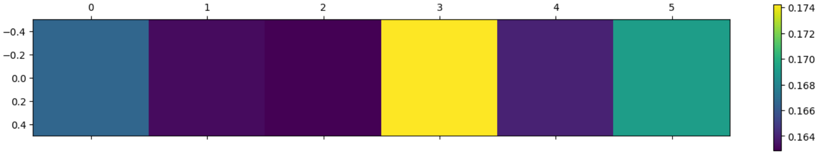

Figure 3: Large values of $\boldsymbol{\beta} = [1000, 1000]^\top$ promote uniformity $(1/N)$.

Making Matrices from these Random Vectors

Definition (Beta/ Dirichlet Random Covariance Matrices[4])

Now that we are able to sample vectors in $\mathbb{R}^t$ from the Beta and Dirichlet distribution, we can stack these vectors on top of each other N times to get an $N \times T$ data matrix $\mathbf{H}$. This makes *Beta Random Matrices* or *Dirichlet Random Matrices*. To observe the spectrum, we construct the covariance matrix to make *Beta Random Covariance Matrices* (BRCM) or *Dirichlet Random Covariance Matrices* (DRCM).

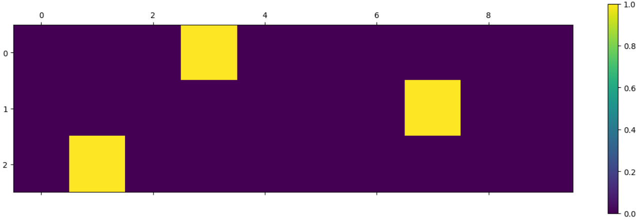

Figure 4: The DRCM is constructed by $\frac{1}{10}\mathbf{A}\mathbf{A}^\top$, where $\mathbf{A}$ is seen above.

Spectrum of the Beta Random Matrices

Conjecture

The spectrum of the BRCM, for arbitrary $\boldsymbol{\beta} = [\beta_1, \beta_2]^\top$, is of Marčenko-Pastur type.

Definition (Sub-Gaussian variables)

A random variable X with $\mu = \mathbb{E}[X] < \infty$ is *sub-Gaussian* if $\exists \sigma > 0$ such that

$$\mathbb{E}[\exp(\lambda(X - \mu))] \leq \exp\left(\frac{\lambda^2\sigma^2}{2}\right), \quad \forall \lambda \in \mathbb{R}$$

Lemma (Marchal, Arbel, 2017[5])

The Beta(β, β) distribution is strictly sub-Gaussian.

Spectrum of the Beta Random Matrices (cont.)

Proof of Lemma.

First, observe that the j^{th} moment of Beta(β_1, β_2) for a random variable X is given by:

$$\mathbb{E}[X^j] = \frac{(\beta_1)_j}{(\beta_1+\beta_2)_j}, \quad \mathbb{E}[X] = \frac{\beta_1}{\beta_1+\beta_2}, \mathbb{V}[X] = \frac{\beta_1\beta_2}{(\beta_1+\beta_2)^2(\beta_1+\beta_2+1)}$$

Now letting $\beta_1 = \beta_2, \sigma^2(\boldsymbol{\beta}) = \mathbb{V}[\text{Beta}(\beta,\beta)] = 1/(4(2\beta+1))$. Also since X is symmetric around $\frac{1}{2}$, then the even moments are non-zero.

$$\mathbb{E}\left[\exp\left(X-\frac{1}{2}\right)\right] = \sum_{j=0}^{\infty} \mathbb{E}\left[(X-1/2)^{2j}\right]\frac{1}{(2j)!}$$

$$\mathbb{E}\left[\left(X-\frac{1}{2}\right)^{2j}\right]\frac{1}{(2j)!} = \frac{1}{2^{2j}j!}\frac{(\beta)_j}{(2\beta)_{2j}}$$

$$\leq \frac{1}{2^{2j}j!}\frac{1}{(2(2\beta+1))^j} = \frac{\sigma^{2j}(\boldsymbol{\beta})}{2^j j!}$$

□

More Results on the Spectrum of BRCM

Theorem (Vershynin, 2011(Thm 5.39)[6])

Let $\mathbf{A}$ *be an* $N \times n$ *matrix are independent sub-gaussian isotropic random vectors in* $\mathbb{R}^n$. *Then for every* $t \geq 0$ *with probability at least* $1 - 2\exp(-ct^2)$ *one has*

$$\sqrt{N} - C\sqrt{n} - t \leq s_{min}(\mathbf{A}) \leq s_{max}(\mathbf{A}) \leq \sqrt{N} + C\sqrt{n} + t. \tag{10}$$

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We can even extend this to non-isotropic distributions, which the Beta distribution is since $\mathbb{E}[Beta(\beta_1, \beta_2)] = \frac{1}{N} > 0$.

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Possible Proof Directions.

Since the Beta distribution is sub-Gaussian by the previous lemma [5] then by [6] we know that the BRCM has limiting spectral distribution which follows the domain of Marčenko-Pastur density with high probability. □

Spectrum of the Dirichlet Random Matrices

Conjecture

The spectrum of the DRCM is of Marčenko-Pastur type.

Theorem (Yaskov, 2016[7])

If $(\mathbf{x}_p^\top \mathbf{A}_p \mathbf{x}_p - \mathrm{tr}\{\mathbf{A}_p)\})/p \xrightarrow{p} 0$ *as* $p \to \infty$ *for all sequences of* $p \times p$ *complex matrices* $\mathbf{A}_p$ *with uniformly bounded spectral norms* $\|\mathbf{A}_p\|$, *then the spectrum converges weakly to Marčenko-Pastur with probability 1.*

Spectrum of the Dirichlet Random Matrices (cont.)

Proof.

We will use the Cauchy–Stieltjes transform method. By the Stieltjes continuity theorem, it suffices to prove that $s_n(z) \to s(z)$ a.s. $\forall z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$, where $s_n(z)$ and $s(z)$ are the Stieltjes transforms of μ and μ_{MP}, respectively

$$s_n(z) = \int_{\mathbb{R}} \frac{\mu(\mathrm{d}\lambda)}{\lambda - z} \quad \text{and} \quad s_(z) = \int_{\mathbb{R}} \frac{\mu_{\mathrm{MP}}(\mathrm{d}\lambda)}{\lambda - z}$$

Since μ is isotropic, $s_n(z) = \mathrm{tr}\left(n^{-1}\mathbf{X}\mathbf{X}^\top - z\mathbf{I}_p\right)^{-1}/p$. Now fix $z \in \mathbb{C}$ with $\mathrm{Im}(z) = v > 0$, then through a Martingale type argument, $s_n(z) - \mathbb{E}s_n(z) \to 0$ a.s.. Lastly through a Sherman-Morrison-Woodbury argument we arrive at $\mathbb{E}s_n(z) \to s(z)$. □

Spectrum of the Dirichlet Random Matrices (cont.)

Possible Proof Direction.

By [7], we need to prove that for $\mathbf{A}$, a Dirichlet Random Matrix, that the DRCM $\mathbf{C} = \mathbf{A}^\top \mathbf{A}$ has bounded operator norm. Since $\mathbf{C}_{ij} > 0, \in \mathbb{R}$, then by Perron-Frobenius we know that $\lambda_{\max} \leq \max_i \sum_j \mathbf{C}_{ij} < \infty$, so $\|\mathbf{C}\|_2$ is finite, and then all induced norms are bounded.

Next we need to show that, $\mathbb{V}[\mathbf{x}^\top \mathbf{C}\mathbf{x}/p] \to 0$. Consider $\mathbf{x} = \vec{\mathbf{1}}$, then since each row is rescaled to sum to 1, then we would have $\mathbf{x}^\top \mathbf{x} = p$. Since we are dividing by p, the variance of any constance is 0, so by Theorem, we have DRCM are of Marčenko-Pastur type. □

Numerical Results in the Bulk

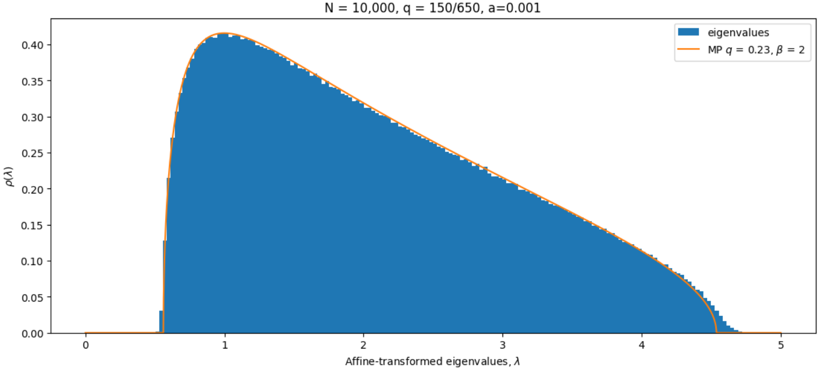

Figure 5: DRCM's spectrum converge to Marčenko-Pastur with hand tuned σ.

Numerical Results on the Edge

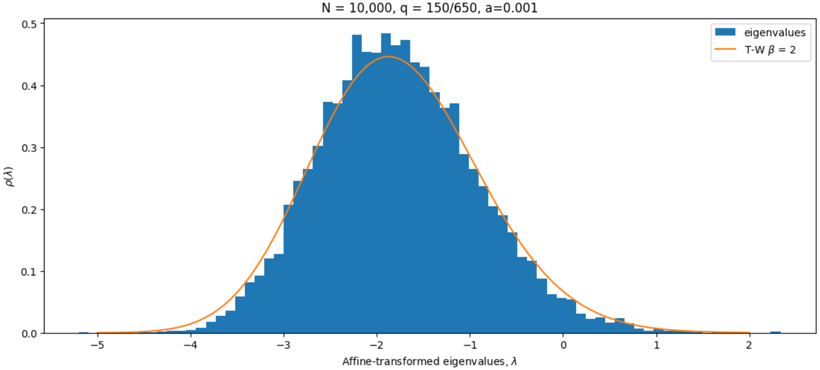

Figure 6: DRCM appears to have a Tracy-Widom like decay on the edge.

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- examimed the Beta and Dirichlet distribution, exploiting them to make RM, and
- conjectured Marčenko-Pastur spectrum from both classes, numerically verifying it.

Future Directions

Theoretic Considerations:

- The Beta and Dirichlet matrices have $\mathbb{E}(x_{ij}) \neq 0$, so I am trying to find a fix for these, either through a $\mathbf{Y} = \mathbf{W}^{1/2}\mathbf{X}$ or reworking the proof.

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- Continue to compare different values of $\dim(\boldsymbol{\beta})$, (namely compare Beta and 2-Dirichlet)

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Future Directions

Theoretic Considerations:

- The Beta and Dirichlet matrices have $\mathbb{E}(x_{ij}) \neq 0$, so I am trying to find a fix for these, either through a $\mathbf{Y} = \mathbf{W}^{1/2}\mathbf{X}$ or reworking the proof.
- Use Markov style arguments and adjust the proof using [8, 9]
- Determine the point mass at $x = 0$.

Numerical Considerations:

- Even with proof that these converge to Marčenko-Pastur distributions, the exact constants in the law aren't known.
- I have hand tuned the σ parameter, but I would like to solve $\sigma = \sigma(q, \boldsymbol{\beta}, \dim(\boldsymbol{\beta}), \beta_i)$.
- Continue to compare different values of $\dim(\boldsymbol{\beta})$, (namely compare Beta and 2-Dirichlet)
- Let $\boldsymbol{\beta}$ change from entry to entry.

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