Block Subset Selection based on Randomized QR with Column Pivoting

A Tensor $\star_{\mathrm{M}}$ Product Framework

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Motivation: Optimal Experiment Design

Goal: Find the best way to acquire more data, i.e. optimal experiment design (OED).

Definition (Fisher Information Matrix (FIM))

If there are m model parameters in experiment design c, with parameters $\boldsymbol{\theta} = \begin{pmatrix} \theta_1 & \theta_2 & \cdots & \theta_m \end{pmatrix}^\top$, and $f(X, \boldsymbol{\theta})$ is a probability density function, then the Fisher Information Matrix becomes

$$[\mathbf{F}(c, \boldsymbol{\theta})]_{i,j} = -\mathbb{E}\left[\left(\frac{\partial}{\partial \theta_i} \log f(X; \boldsymbol{\theta}) \right) \left(\frac{\partial}{\partial \theta_j} \log f(X; \boldsymbol{\theta}) \right) \middle| \, \boldsymbol{\theta} \right] \tag{1}$$

If data is collected under design c with parameters $\boldsymbol{\theta}$, we want to solve

$$\max_{c \in \mathcal{S}} \phi\left(\mathbf{F}(c, \boldsymbol{\theta})\right) = \max_{c \in \mathcal{S}} \frac{1}{2} \log \det \mathbf{F}(c, \boldsymbol{\theta}) \tag{2}$$

over all designs in the design space, $\mathcal{S} \subset \mathcal{C}$.

(Block) Volume is important!

Definition (volume)

The *volume* of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the product of the singular values:

$$\text{vol}(\mathbf{A}) = \prod_{j=1}^{\min(m,n)} \sigma_j(\mathbf{A}), \quad \text{and} \quad \textit{if } m = n, \text{vol}(\mathbf{A}) = |\det(\mathbf{A})| \tag{3}$$

Under statistic assumptions,

$$\mathbf{F} = \begin{bmatrix} \mathbf{J}^\top & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{J} \\ \mathbf{I} \end{bmatrix} = \mathbf{J}^\top \mathbf{J} + \mathbf{I} \quad \Longrightarrow \quad \log\det \mathbf{F} = \log\det\left(\mathbf{J}^\top \mathbf{J} + \mathbf{I}\right)$$

We **must** select a principal submatrix that respect block structure.

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Block Column Pivoted Householder QR: Computing $\mathbf{AP} = \mathbf{QR}$

function `col_block_pivoted_qr`$(\mathbf{A}, k)$
$\mathbf{W} = \texttt{copy}(\mathbf{A})$ ▷ $\mathbf{A}$ is $m \times (nb)$, require $k \leq \min\{m/b, n\}$
$\pi = (1, \dots, n)$
$i = 1$ ▷ i indicates the i^{th} column block
while $i \leq k$ **do**
  $p = \texttt{find\_pivot}(\mathbf{W}(:, i{:}n))$ ▷ find and apply the pivot
  $p = p + i - 1$
  $\mathbf{W}(:, [i, p]) = \mathbf{W}(:, [p, i])$
  $\pi([i, p]) = \pi([p, i])$ ▷ update $\mathbf{W}$ and i
  $[\mathbf{V}, \sim] = \texttt{qr}(\mathbf{W}(:, i), \texttt{"econ"})$
  $\mathbf{W}(:, i) = \mathbf{V}$
  $i = i + 1$
  $\mathbf{W}(:, i{:}n) = (\mathbf{I}_m - \mathbf{V}\mathbf{V}^\top)\mathbf{W}(:, i{:}n)$
end while
$\mathbf{Q} = \mathbf{W}(:, i{:}k)$
$\mathbf{R} = \mathbf{Q}^\top \mathbf{A}(:, \pi)$
return $(\mathbf{Q}, \mathbf{R}, \pi)$

Connection between Volume, Singular Values, and Rank Revealers

Definition (Rank-revealer [1])

If there is a $\mu_{m,n,k} \in \text{poly}(m, n, k)$ such that for any $\mathbf{A} \in \mathbb{R}^{m \times n}$ and any $1 \leq k \leq \min(m, n)$, the *rank revealer* computes a rank $\leq k$ matrix $\mathbf{A}_k$ such that

$$\frac{1}{\mu_{m,n,k}} \sigma_j(\mathbf{A}) \leq \sigma_j(\mathbf{A}_k) \leq \mu_{m,n,k} \sigma_j(\mathbf{A}), \quad 1 \leq j \leq k \tag{4}$$

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and

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Truncated SVD:

- rank-revealer:
  $\mu = 1$ ☺

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- practical: $\mathcal{O}(kmn)$ ☺

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ν-Near local max vol QR [1]:

- rank-revealer: $\mu = \sqrt{1 + 5\nu^2 kn}$ ☺

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Column Pivoted QR [2]:

- practical: $\mathcal{O}(kmn)$ ☺
- not rank-revealer: $\mu = 2^k\sqrt{n-k}$ ☹

ν-Near local max vol QR [1]:

- rank-revealer: $\mu = \sqrt{1+5\nu^2 kn}$ ☺
- fast: $\mathcal{O}\left(\sqrt{kn}\right)$ ☺

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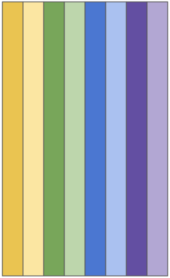
This ν-max vol framework is inadequate for our setting: a toy example

Let $\mathbf{A} = [\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3\ \mathbf{A}_4]$, where $\mathbf{A}_i \in \mathbb{R}^{m\times 2}$, $\|\mathbf{A}_i(:,1)\| = 1$, $\|\mathbf{A}_i(:,2)\| = \epsilon$, for all $i = 1, \ldots, 4$.

Goal: choose three blocks (six rows) such that we maximize $\mathrm{vol}([\mathbf{A}_{\pi(1)}\ \mathbf{A}_{\pi(2)}\ \mathbf{A}_{\pi(3)}])$



Block matrix

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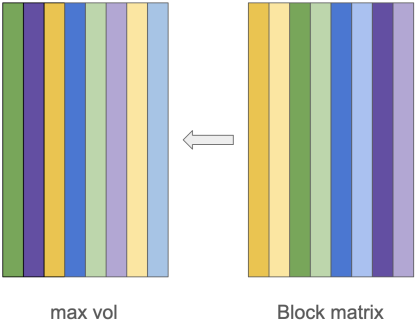

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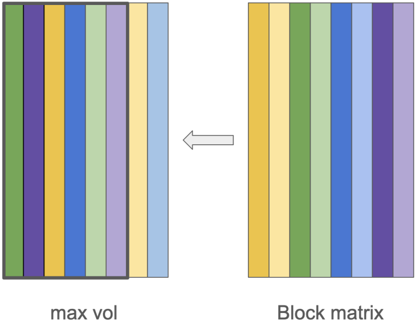

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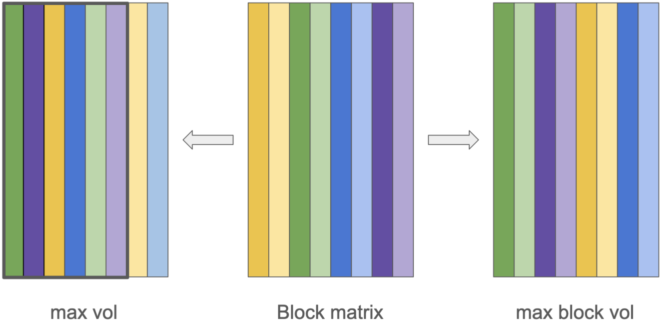

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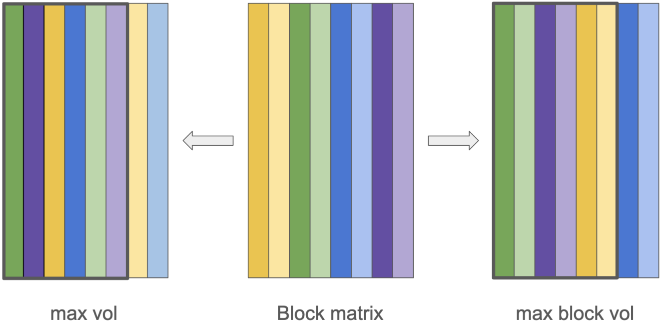

Greedy OED as block pivoted QR

Let $b \geq 2$ be the block size. Define the helper function `expand` such that

$$\texttt{expand([]) = []} \qquad \text{and} \qquad \texttt{expand}([\texttt{i}] + \mathcal{S}) = \texttt{i}b : (\texttt{i}+1)b + \texttt{expand}(\mathcal{S})$$

Fact: If $[\sim, R_\mathcal{S}] = \texttt{qr}\left(\begin{bmatrix} \mathbf{J}_\mathcal{C}^\top \\ \mathbf{I}_{|\mathcal{C}|b} \end{bmatrix}\left(:, \texttt{expand}(\mathcal{S})\right)\right)$, then $\log\det(\mathbf{F}(c, \boldsymbol{\theta})) = \log\det(R_\mathcal{S})$

Idea: Define a suitable `block_pivoted_qr` function that we will call as

$$[\sim, R, \texttt{piv}] = \texttt{block\_pivoted\_qr}\left(b, \begin{bmatrix} \mathbf{J}_\mathcal{C}^\top \\ \mathbf{I}_{|\mathcal{C}|b} \end{bmatrix}\right),$$

and use $\mathcal{C}_{\text{ranked}} = \texttt{expand}^{-1}(\texttt{piv})$. Truncate $\mathcal{C}_{\text{ranked}}$ to get $\mathcal{S}$.

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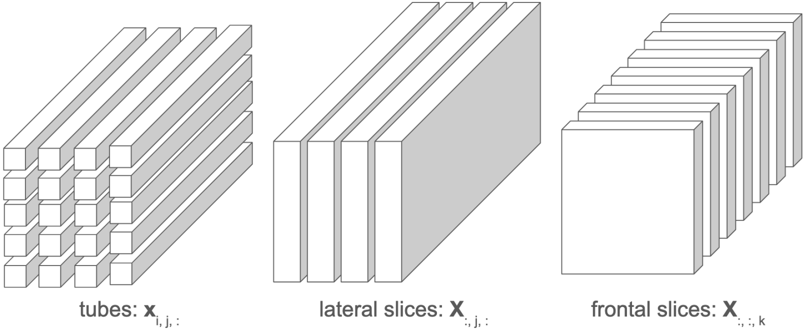
Tensors Terminology



See [Kolda and Bader, 2009] for more great figures!

The $\star_{\mathbf{M}}$ Product

Transform Domain
Let $\hat{\mathcal{T}} \in \mathbb{R}^{m\times n\times b}$ with i, j^{th} tube equal to the product of the i, j^{th} tube of tensor $\mathcal{T} \in \mathbb{R}^{m\times n\times b}$ with matrix $\mathbf{M} \in \mathbb{R}^{b\times b}$ by $\hat{\mathcal{T}} = \mathcal{T} \times_3 \mathbf{M}$,

$$(\mathcal{T} \times_3 \mathbf{M})(i, j, :) = \mathbf{M}\mathcal{T}(i, j, :).$$

Facewise-product
Here, the frontal facewise-product is defined in frontal slice k as

$$(\mathcal{T}_1 \Delta \mathcal{T}_2)(:, :, k) = \mathcal{T}_1(:, :, k)\mathcal{T}_2(:, :, k).$$

Definition ($\star_{\mathbf{M}}$ product [4, 5])

The $\star_{\mathbf{M}}$ *product* for invertible $\mathbf{M}$ is defined as

$$\mathcal{A} \star_{\mathbf{M}} \mathcal{B} := ((\mathcal{A} \times_3 \mathbf{M})\Delta(\mathcal{B} \times_3 \mathbf{M})) \times_3 \mathbf{M}^{-1}. \tag{6}$$

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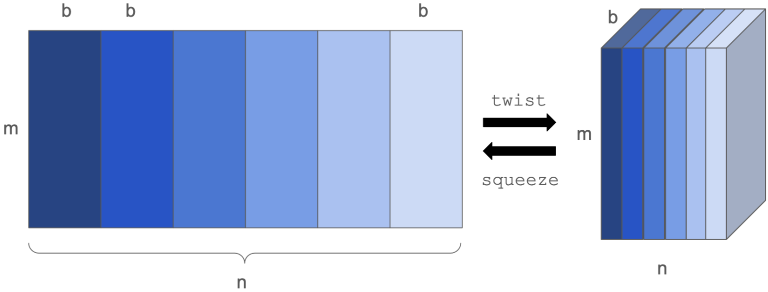
Block setting arises naturally in OED

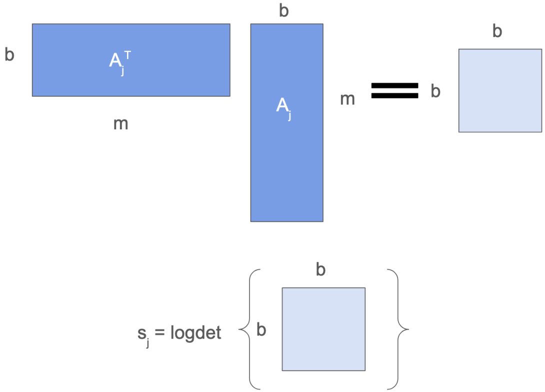
We are concerned with blocks because

- m is the number of model parameters,
- b is the time series data, and
- n is the number of distinct experiments run.



Determine the “most important” prior experiment

Initialize the permutation $\pi = (1, \cdots, n)$.
For each $1 \leq j \leq n$, compute $s_j = \log\det\left(\mathbf{A}_j^\top \mathbf{A}_j\right)$.

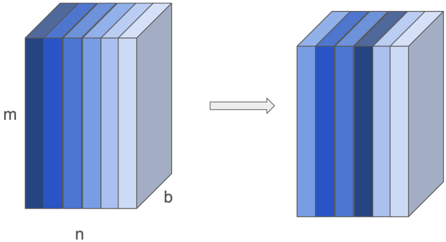


Then $k = \arg\max\{s_1, \cdots, s_n\}$.

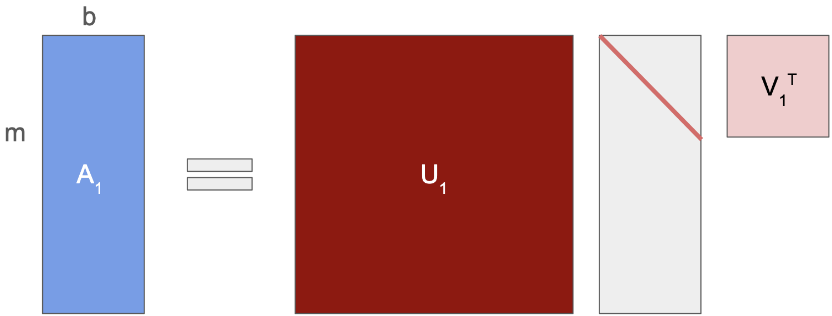
Permute the highest score'ed experiment to the front.

Since $k = \arg\max\{s_1, \cdots, s_n\}$, we let

- $\mathcal{A}(:, 1, :) \leftrightarrow \mathcal{A}(:, k, :)$ and
- $\pi(1) \leftrightarrow \pi(k)$



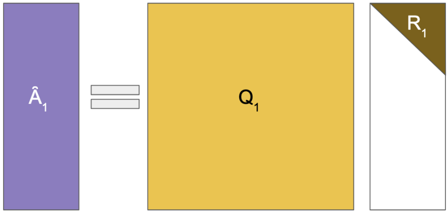
Obtain the right singular vectors of the best experiment



Update $\mathcal{A} \leftarrow \mathcal{A} \times_3 \mathbf{V}_1$

This is equivalent to lateral slice updates $\mathbf{A}_j \leftarrow \mathbf{A}_j \mathbf{V}_1^\top$

Compute the matrix QR on the transformed first lateral slice



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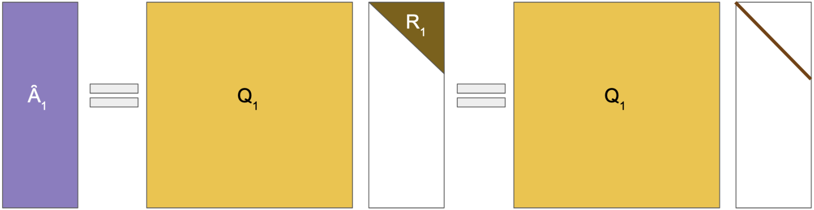
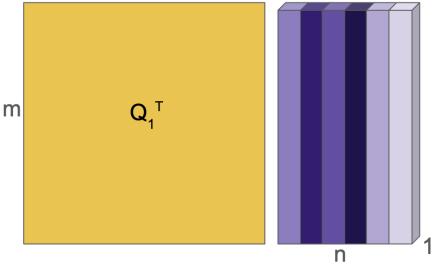


Update the frontal slices

With access to $\mathbf{Q}_1^\top$, we perform $\mathcal{A}(:,:,\ell) \leftarrow \mathbf{Q}_1^\top \mathcal{A}(:,:,\ell)$ for all $\ell \in \{1,\ldots,b\}$



This is equivalent to $\mathbf{A}_j \leftarrow \mathbf{Q}^\top \mathbf{A}_j$

The $\star_{\mathbf{M}}$ product Decomposition

Let $\mathcal{A}^{(1)}$ denote the value of $\mathcal{A}$ on *input* to the above procedure. By the end of the procedure we have a decomposition of the form

$$\mathcal{A}^{(1)} \star_{\mathbf{V}_1} \mathcal{P}^{(1)} = \mathcal{Q}^{(1)} \star_{\mathbf{V}_1} \mathcal{R}^{(1)}, \tag{7}$$

where $\mathcal{P}^{(1)}$, $\mathcal{Q}^{(1)}$, and $\mathcal{R}^{(1)}$ are defined implicitly; using π and $\mathbf{V}_1$ as they exist at step 3, and using $\mathbf{Q}_1$ from step 5, we have

$$[\mathcal{P}^{(1)} \times_3 \mathbf{V}_1](:,:,\ell) = \mathbf{I}_n(:,\pi) \tag{8}$$

$$[\mathcal{Q}^{(1)} \times_3 \mathbf{V}_1](:,:,\ell) = \mathbf{Q}_1 \tag{9}$$

$$[\mathcal{R}^{(1)} \times_3 \mathbf{V}_1](:,:,\ell) = \mathbf{Q}_1^\top [\mathcal{A}^{(1)} \times_3 \mathbf{V}_1](:,:,\ell)\mathbf{I}_n(:,\pi), \tag{10}$$

for each $\ell \in \{1, \dots, p\}$.

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The $\star_{\mathbf{M}}$ Connection to Column Block Pivoted QR

Consider the $m \times bn$ unfolding $\mathbf{A} := [\mathbf{A}_j]_{j=1}^{n}$ of $\mathcal{A}$.

function `col_block_pivoted_qr`$(\mathbf{A}, k)$
$\mathbf{W} = \texttt{copy}(\mathbf{A})$
$\pi = (1, \dots, n)$
$i = 1$
while $i \leq k$ **do**
  $p = \texttt{find\_pivot}(\mathbf{W}(:, i{:}n))$
  $p = p + i - 1$
  $\mathbf{W}(:, [i, p]) = \mathbf{W}(:, [p, i])$
  $\pi([i, p]) = \pi([p, i])$
  $[\mathbf{V}, \sim] = \texttt{qr}(\mathbf{W}(:, i), \texttt{"econ"})$
  $\mathbf{W}(:, i) = \mathbf{V}$
  $i = i + 1$
  $\mathbf{W}(:, i{:}n) = (\mathbf{I}_m - \mathbf{V}\mathbf{V}^\top)\mathbf{W}(:, i{:}n)$
end while
$\mathbf{Q} = \mathbf{W}(:, i{:}k)$
$\mathbf{R} = \mathbf{Q}^\top \mathbf{A}(:, \pi)$
return $(\mathbf{Q}, \mathbf{R}, \pi)$

The $\star_{\mathbf{M}}$ Connection to Column Block Pivoted QR

1: **function** `col_block_pivoted_qr`$(\mathbf{A}, k)$
2: $\mathbf{W} =$ `copy`$(\mathbf{A})$
3: $\pi = (1, \ldots, n)$
4: $i = 1$
5: **while** $i \leq k$ **do**
6: $\quad p =$ `find_pivot`$(\mathbf{W}(:, i{:}n))$
7: $\quad p = p + i - 1$
8: $\quad \mathbf{W}(:, [i, p]) = \mathbf{W}(:, [p, i])$
9: $\quad \pi([i, p]) = \pi([p, i])$
10: $\quad [\mathbf{V}, \sim] =$ `qr`$(\mathbf{W}(:, i),$ `"econ"`$)$
11: $\quad \mathbf{W}(:, i) = \mathbf{V}$
12: $\quad i = i + 1$
13: $\quad \mathbf{W}(:, i{:}n) = (\mathbf{I}_m - \mathbf{V}\mathbf{V}^\top)\mathbf{W}(:, i{:}n)$
14: **end while**
15: $\mathbf{Q} = \mathbf{W}(:, i{:}k)$
16: $\mathbf{R} = \mathbf{Q}^\top \mathbf{A}(:, \pi)$
17: **return** $(\mathbf{Q}, \mathbf{R}, \pi)$

Consider the $m \times nb$ unfolding $\mathbf{A} := [\mathbf{A}_j]_{j=1}^{n}$ of $\mathcal{A}$.

- Lines 8 and 9 are equivalent to right-multiplying $\mathbf{A}$ by $\mathbf{I}_n(:, \pi) \otimes \mathbf{I}_b$.

The $\star_{\mathbf{M}}$ Connection to Column Block Pivoted QR

1: **function** `col_block_pivoted_qr`$(\mathbf{A}, k)$
2: $\mathbf{W} =$ `copy`$(\mathbf{A})$
3: $\pi = (1, \ldots, n)$
4: $i = 1$
5: **while** $i \leq k$ **do**
6: $\quad p =$ `find_pivot`$(\mathbf{W}(:, i{:}n))$
7: $\quad p = p + i - 1$
8: $\quad \mathbf{W}(:, [i, p]) = \mathbf{W}(:, [p, i])$
9: $\quad \pi([i, p]) = \pi([p, i])$
10: $\quad [\mathbf{V}, \sim] =$ `qr`$(\mathbf{W}(:, i),$ `"econ"`$)$
11: $\quad \mathbf{W}(:, i) = \mathbf{V}$
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- Lines 8 and 9 are equivalent to right-multiplying $\mathbf{A}$ by $\mathbf{I}_n(:, \pi) \otimes \mathbf{I}_b$.
- Line 13 is equivalent to right-multiplying $\mathbf{A}$ by $\mathbf{I}_n \otimes \mathbf{V}_1^\top$.

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The $\star_{\mathbf{M}}$ Connection to Column Block Pivoted QR

```
1:  function col_block_pivoted_qr(A, k)
2:  W = copy(A)
3:  π = (1, ..., n)
4:  i = 1
5:  while i ≤ k do
6:      p = find_pivot(W(:, i:n))
7:      p = p + i − 1
8:      W(:, [i, p]) = W(:, [p, i])
9:      π([i, p]) = π([p, i])
10:     [V, ~] = qr(W(:, i), "econ")
11:     W(:, i) = V
12:     i = i + 1
13:     W(:, i:n) = (I_m − VV^T)W(:, i:n)
14: end while
15: Q = W(:, i:k)
16: R = Q^T A(:, π)
17: return (Q, R, π)
```

Consider the $m \times bn$ unfolding $\mathbf{A} := [\mathbf{A}_j]_{j=1}^{n}$ of $\mathcal{A}$.

- Lines 8 and 9 are equivalent to right-multiplying $\mathbf{A}$ by $\mathbf{I}_n(:, \pi) \otimes \mathbf{I}_b$.
- Step 4 is equivalent to right-multiplying $\mathbf{A}$ by $\mathbf{I}_n \otimes \mathbf{V}_1^{\top}$.
- Line 16 is equivalent to left-multiplying $\mathbf{A}$ by $\mathbf{Q}_1^{\top}$.

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Consider the $m \times bn$ unfolding $\mathbf{A} := [\mathbf{A}_j]_{j=1}^n$ of $\mathcal{A}$.

- Lines 8 and 9 are equivalent to right-multiplying $\mathbf{A}$ by $\mathbf{I}_n(:, \pi) \otimes \mathbf{I}_b$.
- Step 4 is equivalent to right-multiplying $\mathbf{A}$ by $\mathbf{I}_n \otimes \mathbf{V}_1^\top$.
- Line 16 is equivalent to left-multiplying $\mathbf{A}$ by $\mathbf{Q}_1^\top$.

We recurse on the $(m-b) \times (nb-b)$ submatrix $\mathbf{A}((b+1){:}m, (b+1){:}nb)$.

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Randomization can help!

The matrix $\mathbf{A} = \begin{bmatrix} \mathbf{J}_{\mathcal{C}}^\top \\ \mathbf{I}_{|\mathcal{C}|b} \end{bmatrix}$ is large!

Suppose we know $\mathbf{J}_{\mathcal{C}}$ has p columns, and we want to narrow down to k experiments.

For a tuning parameter $\gamma \in [1, 2]$, set $d = \lceil \gamma b k \rceil$. Our sketching matrix becomes

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\Omega}_1 & \boldsymbol{\Omega}_2, \end{bmatrix} \tag{11}$$

where $\boldsymbol{\Omega}_1 \in \mathbb{R}^{d \times p}$ and $\boldsymbol{\Omega}_2 \in \mathbb{R}^{d \times |\mathcal{C}|b}$ are i.i.d. mean zero unit variance Gaussian entries. Instead of running the algorithm on $\mathbf{A}$, instead we can make pivot decisions on $\boldsymbol{\Omega}\mathbf{A}$

$$\boldsymbol{\Omega}\mathbf{A} = \boldsymbol{\Omega}_1 \mathbf{J}_{\mathcal{C}}^\top + \boldsymbol{\Omega}_2$$

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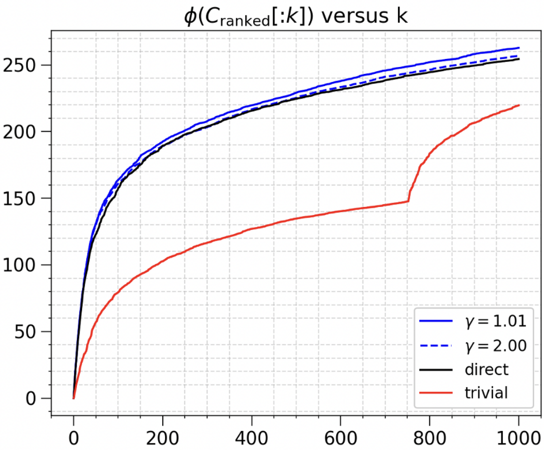
Numerical Experiment: Choosing Quantum Gates Efficiently



Use randLAPACK's randomized block QRCP algorithm to make 1000 block pivot decisions.

Total time for block pivoted QR on sketch: **36 sec**

Total time for each input matrix:

- **SA** (direct): 48 minutes
- $\mathbf{S\Omega_{2.00}A}$: 18 minutes
- $\mathbf{S\Omega_{1.01}A}$: 7 minutes

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- and sped up the optimization problem.

Open questions:

- max-volume pivoting is NP-hard, so what greedy heuristic should we use?
- can we quantify the suboptimality of the greedy method? sketched greedy method?

Questions?

Thank You!

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References

[1] Anil Damle et al. *How to reveal the rank of a matrix?* arXiv:2405.04330 [math]. June 2024. DOI: 10.48550/arXiv.2405.04330. URL: http://arxiv.org/abs/2405.04330 (visited on 10/18/2025).

[2] Peter Businger and Gene H. Golub. "Linear least squares solutions by householder transformations". en. In: *Numerische Mathematik* 7.3 (June 1965), pp. 269–276. ISSN: 0945-3245. DOI: 10.1007/BF01436084. URL: https://doi.org/10.1007/BF01436084 (visited on 10/04/2025).

[3] Tamara G. Kolda and Brett W. Bader. "Tensor Decompositions and Applications". In: *SIAM Review* 51.3 (Aug. 2009), pp. 455–500. ISSN: 0036-1445. DOI: 10.1137/07070111X.

[4] M. E. Kilmer and C. D. Martin. "Factorization strategies for third-order tensors". In: *Linear Algebra Appl.* 435.3 (2011), pp. 641–658.

[5] Misha E Kilmer et al. "Tensor-tensor algebra for optimal representation and compression of multiway data". In: *Proceedings of the National Academy of Sciences* 118.28 (2021), e2015851118.