Real and Complex Analysis Qualifying Exam: Study Guide

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1 Introduction

This document was prepared as a culmination of my notes and study materials for my institution's analysis qualifying examination. The topics presented below are based off of what a first year math PhD student would learn in the graduate level Real and Complex analysis sequence. However, each institution and professor highlights slightly different topics so there might be some slight variation. I also claim no novelty in this document. All of the theorems, lemmas, definitions, etc, should be cited from the standard analysis books, and the subsequent proofs are the formulation presented from the respective citation. The only thing that this document does is it condenses all of a standard real and complex analysis course into one file for studying purposes. The notation, while at times contradictory, is all true to the source from which it was taken. There are some short comments interspersed between the theorems that are meant to serve as a slight outline or transistion from one topic to another. Lastly, I am sure there are typos in this document, and when one is found, it will be updated. Please feel free to make me aware of any typos that you see.

2 Complex Analysis

2.1 Background Material

Definition 2.1 (Primative [3]). A function f has a *primitive* if there exists a function F that is holomorphic and whose derivative is precisely f.

Definition 2.2 (Conformal Equivalence [2]). We call two regions Ω_1 and Ω_2 *conformal equivalence* if there exists a $\varphi \in H(\Omega_1)$ such that φ is one-to-one in Ω_1 and such that $\varphi(\Omega_1) = \Omega_2$, i.e., if there exists a conformal one-to-one mapping of Ω_1 onto Ω_2. Under these conditions, the inverse of φ is holomorphic in Ω_2, and hence is a conformal mapping of Ω_2 onto Ω_1.

2.2 Core Results

Theorem 2.1 (Goursat's Theorem [3]). *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)\,\mathrm{d}z = 0$$

whenever f is holomorphic in Ω.

Theorem 2.2 (Cauchy's Theorem for a Triangle [2]). *Suppose Δ is a closed triangle in a plane open set Ω, $p \in \Omega$, and $f \in H(\Omega \setminus \{p\})$. Then*

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0$$

.

Proof. We assume first that $p \notin \Delta$. Let a, b, and c be the vertices of Δ, and let a', b', c' be the midpoints of $[b, c], [c, a]$, and $[a, b]$, respectively, and consider the four triangles

$$\{a, c', b, \}, \{b, a', c'\}, \{c, b', a'\}, \{a', b', c'\}.$$

If J is the value of the integral $\int_{\partial\Delta} f \, \mathrm{d}z$, it follows that

$$J = \sum_{j=1}^{4} \int_{\partial\Delta^j} f(z) \, \mathrm{d}z \, .$$

The absolute value of at least one of the integrals on the right hand side is therefore at least $|J/4|$. Call the corresponding triangle Δ_1, repeat the argument with Δ_1 in place of Δ, and so forth. This generates a sequence of triangles Δ_n such that $\Delta \supset \Delta_1 \supset \Delta_2 \supset \cdots$, such that the length of $\partial\Delta_n$ is $2^{-n}L$ if L is the length of $\partial\Delta$, and such that

$$|J| \leq 4^n \left| \int_{\partial\Delta_n} f(z) \, \mathrm{d}z \right| \qquad (n = 1, 2, 3, ...).$$

There is a (unique) point z_0 wich the triangles Δ_n have in common. Since Δ is compact, $z_0 \in \Delta$, so f is differentiable at z_0. Let $\epsilon > 0$. be given. There exists an $r > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon|z - z_0|$$

whenever $|z - z_0| < r$, and there exists an n such that $|z - z_0| < r$ for all $z \in \Delta_n$. For this n we also have $|z - z_0| \leq 2^{-n}L$ for all $z \in \Delta_n$. By the corollary

$$\int_{\partial\Delta_n} f(z) \, \mathrm{d}z = \int_{\partial\Delta_n} [f(z) - f(z_0) - f'(z_0)(z - z_0)] \, \mathrm{d}z \, ,$$

so that way we can combine with the other intequality implies

$$\left| \int_{\partial\Delta_n} f(z) \, \mathrm{d}z \right| \leq \epsilon(2^{-n}L)^2,$$

and now that shows tht $|J| \leq \epsilon L^2$. Hence $J = 0$ if $p \notin \Delta$. Assume next that p is a vertex of Δ, say $p = a$. If a, b, and c are colinear, then it is trivial for any continuous f by the FTC. If not, choose points $x \in [a, b]$ and $y \in [a, c]$, both close to a, and observe that the integral of f over $\partial\Delta$ is the sum of the integrals over the boundaries of the triangles $\{a, x, y\}, \{x, b, y\}$, and p. Hence the integral over $\partial\Delta$ is the sum of the integrals over $[a, x], [x, y]$, and $[y, a]$, and since these intervals can be made arbitrarily short and f is bounded on Δ, we again obtain that the integral equals zero.
Finally, if p is an arbitrary point of Δ, apply the preceding result to $\{a, b, p\}, \{b, c, p\}$, and $\{c, a, p\}$ to complete the proof. □

Remark 2.1 (Historical Aside)**.** Why do the same theorem have different names between [2] and [3]? The answer is because Cauchy originally published his theorem in 1825, under the additional assumption that f' is continuous. However, Goursat showed that this assumption was redundant, and stated the theorem in the present form.

Theorem 2.3 (Cauchy's Theorem in a Convex Set[2]). *Supposed Ω is a convex open set, $p \in \Omega$, f is continuous on Ω, and $f \in H(\Omega \setminus \{p\})$. Then $f = F'$ for some $F \in H(\Omega)$. Hence*

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for every closed path γ in Ω.

Proof. Fix $a \in \Omega$. Since Ω is convex, Ω contains the straight line interval from a to z for every $z \in \Omega$, so we can define

$$F(z) = \int_{[a,z]} f(\xi)\,\mathrm{d}\xi \qquad (z \in \Omega).$$

For any z and $z_0 \in \Omega$, the triangle with vertices at a, z_0, and z lies in Ω; hence $F(z) - F(z_0)$ is the integral of f over $[z_0, z]$, by Cauchy's Theorem on a triangle. Fixing z_0, we thus obtain,

$$\frac{F(z) - F(z_0)}{z - z_0} - f'(z_0) = \frac{1}{z - z_0} \int_{[z_0,z]} [f(\xi) - f(z_0)\,\mathrm{d}\xi]\,,$$

if $z \neq z_0$. Given $\epsilon > 0$, the continuity of f at z_0 shows that there is a $\delta > 0$ such that $|f(\xi) - f(z_0)|, \epsilon$ if $|\xi - z_0| < \delta$; hence the absolute value of the $\frac{F(z)-F(z_0)}{z-z_0} < \epsilon$ as soon as $|z - z_0| < \delta$. This proves that $f = F'$. In particular, $F \in H(\Omega)$. Now, we see that letting $[\alpha, \beta]$ be the parameter interval of γ, then the fundamental theorem of calculus shows that

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \int_\gamma F'(z)\,\mathrm{d}z \\
&= \int_\alpha^\beta F'(\gamma(t))\gamma'(t)\,\mathrm{d}t \\
&= F(\gamma(\beta)) - F(\gamma(\alpha)) \\
&= 0
\end{aligned}$$

since $\gamma(\beta) = \gamma(\alpha)$. □

Theorem 2.4 (Cauchy Integral Formula/ Cauchy Theorem[2]). *Suppose $f \in H(\Omega)$, where Ω is an arbitrary open set in the complex plane. If Γ is a cycle in Ω that satisfies*

$$\mathit{Ind}_\Gamma(\alpha) = 0 \qquad \text{for every } \alpha \text{ not in } \Omega,$$

then

$$f(z) \cdot \mathit{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\,\mathrm{d}w \qquad \text{for } z \in \Omega \setminus \Gamma^*$$

and

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

If Γ_0 and Γ_1 are cycles in Ω such that

$$\mathit{Ind}_{\Gamma_0}(\alpha) = \mathit{Ind}_{\Gamma_1}(\alpha) \qquad \text{for every } \alpha \text{ not in } \Omega,$$

then

$$\int_{\Gamma_0} f(z)\,\mathrm{d}z = \int_{\Gamma_1} f(z)\,\mathrm{d}z\,.$$

Proof. The function g defined in $\Omega \times \Omega$ by

$$g(z,w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

is continuous in $\Omega \times \Omega$. Hence we can define

$$h(z) = \frac{1}{2\pi i} \int_\Gamma g(z,w)\,\mathrm{d}w \qquad (z \in \Omega).$$

For $z \in \Omega \setminus \Gamma^*$, the Cauchy formula (2.4) is clearly equivalent to the assertion that

$$h(z) = 0.$$

To prove that, let us prove that $h \in H(\Omega)$. Note that g is uniformly continuous on every compact subset of $\Omega \times \Omega$. If $z \in \Omega, z_n \in \Omega, z_n \to z$, it follows that $g(z_n, w) \to g(z,w)$ uniformly for $w \in \Gamma^*$ (a compact subset of Ω). Hence $h(z_n) \to h(z)$. This proves that h is continuous in Ω, Let Δ be a closed triangle in Ω. Then

$$\int_{\partial\Delta} h(z)\,\mathrm{d}z = \frac{1}{2\pi i} \int_\Gamma \left(\int_{\partial\Delta} g(z,w)\,\mathrm{d}z \right) \mathrm{d}w\,.$$

For each $w \in \Omega, z \to g(z,w)$ is holomorphic in Ω. (The singularity at $z = w$ is removable.) The inner integral on the right side is therefore 0 for every $w \in \Gamma^*$. Morera's Theorem (defined in Theorem 2.7) shows now that $h \in H(\Omega)$.
Next, we let Ω_1 be the set of all complex numbers z for which $\mathrm{Ind}_\Gamma(z) = 0$, and we define

$$h_1(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z}\,\mathrm{d}w \qquad (z \in \Omega_1).$$

If $z \in \Omega \cap \Omega_1$, the definition of Ω_1 makes it clear that $h_1(z) = h(z)$. Hence there is a function $\varphi \in H(\Omega \cup \Omega_1)$ whose restriction to Ω is h and whose restriction to Ω_1 is h_1.
By our hypothesis that $\mathrm{Ind}_\Gamma(\alpha) = 0$, for every α not in Ω, □

Theorem 2.5 (Analyticity of holomorphic functions[2])**.** *For every open set Ω in the plane, every $f \in H(\Omega)$ is representable by a power series expansion.*

Proof. Suppose $f \in H(\Omega)$ and $D(a,R) \subset \Omega$. If γ is a positively oriented circle with center at a and radius $r < R$, the convexity of $D(a,R)$ allows us to apply Cauchy's Theorem on a convex set; by the fact that $\mathrm{Ind}_\gamma(z) = 1, \forall z \in D(a,r)$, we obtain

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi \qquad (z \in D(a,r)).$$

But now we can apply a theorem that says if the integral is of the above form it is representable by a power series, namely let $X = [0, 2\pi], \varphi = \gamma$, and $\mathrm{d}\mu(t) = f(\gamma(t))\gamma'(t)\,\mathrm{d}t$, and we conclude that there is a sequence $\{c_n\}$ such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \qquad (z \in D(a,r)).$$

□

Remark 2.2 (f is holomorphic iff f is representable by power series)**.** While the above proof just showed that if f is holomorphic, then it is representable by a power series. However, [2] already proved the converse, namely if f is representable by power series, then it is holomorphic. Similarly, if f is holomorphic, then so is $f^{(n)}$, or the n^{th} derivative.

Theorem 2.6 (Liouville's Theorem [2])**.** *Every bounded entire finction is constant.*

Proof. Recall the definition of "entire" is any function that is holomorphic over the entire complex plane. Suppose f is entire, $|f(z)| < M$ for all z, and $f(z) = \sum c_n z^n$ for all z. We know by a special case of Parseval's formula that

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{i\theta})|^2 \, \mathrm{d}\theta \,.$$

This means that we have

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq M^2$$

for all r, which is possible only if $c_n = 0$ for all $n \geq 0$. □

We saw by Cauchy's Theorem for a triangle that if $f \in H(\Omega \setminus \{p\})$ then

$$\int_{\partial\Delta} f(z) \, \mathrm{d}z = 0,$$

but it turns out that this also works the other way

Theorem 2.7 (Morera's Theorem [2])**.** *Suppose that f is a continuous complex function in an open set Ω such that*

$$\int_{\partial\Delta} f(z) \, \mathrm{d}z = 0$$

for every closed triangle $\Delta \subset \Omega$. Then $f \in H(\Omega)$

Proof. Let V be a convex open set in Ω. Using Cauchy's formula for a convex set, we can construct an $F \in H(V)$ such that $F' = f$. Since derivatives of holomorphic functions are holomorphic, we have that $f \in H(V)$, for every convex open $V \subset \Omega$, hence $f \in H(\Omega)$. □

Theorem 2.8 (Zeros of holomorphic functions[2])**.** *Suppose Ω is a region, $f \in H(\Omega)$, and*

$$Z(f) = \{a \in \Omega : f(a) = 0\}.$$

Then either $Z(f) = \Omega$, or $Z(f)$ has no limit points in Ω. In the latter case there corresponds to each $a \in Z(f)$ a unique positive integer $m = m(a)$ such that

$$f(z) = (z - a)^m g(z) \qquad (z \in \Omega),$$

where $g \in H(\Omega)$ and $g(a) \neq 0$; furthermore, $Z(f)$ is at most countable.

Proof. Let A be the set of all limit points of $Z(f)$ in Ω. Since f is continuous, $A \subset Z(f)$. Fix $a \in Z(f)$, and choose $r > 0$ so that $D(a, r) \subset \Omega$. Since f is holomorphic, we can write it as a power series, namely

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \quad (z \in D(a, r)).$$

There are now two possibilities. Either all c_n are 0, in which case $D(a,r) \subset A$ and a is an interior point of A, or there is a smallest integer m [necessarily positive, since $f(a) = 0$] such that $c_m \neq 0$. In that case, define

$$g(z) = \begin{cases} (z-a)^{-m} f(z) & (z \in \Omega \setminus \{a\}) \\ c_m & (z = a). \end{cases}$$

Thus $f(z) = (z-a)^m g(z)$. It is clear that $g \in H(\Omega \setminus \{a\})$. But since f can be representable by a power series, this implies

$$g(z) = \sum_{k=0}^{\infty} c_{m+k}(z-a)^k \qquad (z \in D(a,r)).$$

Hence $g \in H(D(a,r))$, so $g \in H(\Omega)$.
Moreover, $g(a) \neq 0$, and the continuity of g shows that there is a neighborhood of a in which g has no zero. Thus a is an isolated point of $Z(f)$.
If $a \in A$, the first case must therefore occur. So A is open. If $B = \Omega \setminus A$, it is clear from the definition of A as a set of limit points that B is open. This Ω is the union of the disjoint open sets A and B. Since Ω is connected, we have either $A = \Omega$, in which case $Z(f) = \Omega$, or $A = \emptyset$. In the latter case, $Z(f)$ has at most finitely many points in each compact subset of Ω, and since Ω is σ-finite, $Z(f)$ is at most countable. □

Theorem 2.9 (Classification of isolated singularties [2]). *If $a \in \Omega$ and $f \in H(\Omega \setminus \{a\})$, then one of the three following cases must occur:*

1. *f has a removable singularity at a.*
2. *There are complex numbers $c_1, ..., c_m$, where m is a positive integer and $c_m \neq 0$, such that*
$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$
has a removable singularity at a.
3. *If $r > 0$ and $D(a,r) \subset \Omega$, then $f(D'(a,r))$ is dense in the plane.*

In case (b), f is said to have a pole of order m *at a. The function*

$$\sum_{k=1}^{m} c_k (z-a)^{-k},$$

a polynomial in $(z-a)^{-1}$, is called the principal part *of f at a. It is clear in this situation that $|f(z)| \to \infty$ as $z \to a$. In case (c), f is said to have an* essential singularity *at a. A statement equivalent to (c) is that to each complex number w there corresponds a sequence $\{z_n\}$ such that $z_n \to a$ and $f(z_n) \to w$ as $n \to \infty$.*

Proof. Suppose (c) fails. Then there exist $r > 0, \delta > 0$, and a complex number w such that $|f(z) - w| > \delta$ in $D'(a,r)$. Let us write D for $D(a,r)$. and D' for $D'(a,r)$. Define

$$g(z) = \frac{1}{f(z) - w} \qquad (z \in D').$$

Then $g \in H(D')$ and $|g| < 1/\delta$. Then since g is bounded in D' and holomorphic everywhere else, then g has a removable singularity, and g extends to a holomorphic function in D.
If $g(a) \neq 0$, then that shows f is bounded in $D'(a, \rho)$ for some $\rho > 0$. Hence, (a) holds, by the previous theorem.
If g has a zero of order $m \geq 1$, the zeros of holomorphic functions theorem shows that

$$g(z) = (z-a)^m g_1(z) \qquad (z \in D),$$

where $g_1 \in H(D)$ and $g_1(a) \neq 0$. Also g_1 has no zero in D', by (1). Put $h = 1/g_1$ in D. Then $h \in H(D), h$ has no zeros in D, and

$$f(z) - w = (z-a)^{-m} h(z) \qquad (z \in D').$$

But h has an expansion of the form

$$h(z) = \sum_{n=0}^{\infty} b_n (z-a)^n \qquad (z \in D),$$

with $b_0 \neq 0$. Now writing $f(z) - w$ that way shows that (b) holds, with $c_k = b_{m-k}, k = 1, ..., m$. □

Theorem 2.10 (Residue Theorem [2]). *Suppose f is a meromorphic function in Ω. Let A be the set of points in Ω at which f has poles. If Γ is a cycle in $\Omega \setminus A$ such that*

$$Ind_\Gamma(\alpha) = 0 \qquad \textit{for all} \qquad \alpha \notin \Omega,$$

then

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, \mathrm{d}z = \sum_{a \in A} Res(f; a) Ind_\Gamma(a).$$

Proof. Let $B = \{a \in A : \mathrm{Ind}_\Gamma(a) \neq 0\}$. Let W be the complement of Γ^*. Then $\mathrm{Ind}_\Gamma(z)$ is constant in each component V of W. If V is unbounded, or if V intersects Ω^C, Then $\mathrm{Ind}_\Gamma(z) = 0$ for every $z \in V$. Since A has no limit point in Ω, we conclude that B is a finite set.
The sum of residues, though formally infinite, is therefore actually finite.
Let $a_1, ..., a_n$ be the points of B, let $Q_1, ..., Q_n$ be the principal parts of f at $a_1, ..., a_n$, and put $g = f - (Q_1 + \cdots + Q_n)$. (If $B = \emptyset$, a possibily is not excluded, then $g = f$.) Put $\Omega_0 = \Omega \setminus (A \setminus B)$. Since g has removable singularities at $a_1, ..., a_n$, then Cauchy's theorem (Theorem 2.4) applied to the function f and the open set Ω_0, shows that

$$\int_\Gamma g(z)\, \mathrm{d}z = 0.$$

Hence

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, \mathrm{d}z = \sum_{i=1}^{n} \frac{1}{2\pi i} \int_\Gamma Q_k(z)\, \mathrm{d}z = \sum_{k=1}^{n} \mathrm{Res}(Q_k; a_k) \mathrm{Ind}_\Gamma(a_k),$$

and since f and Q_k have the same residue at a_k, we obtain the sum. □

Theorem 2.11 (Rouche's Theorem [2]). *Suppose γ is a closed path in a region Ω, such that $Ind_\gamma(\alpha) = 0$ for every α not in Ω. Suppose also that $Ind_\gamma(\alpha) = 0$ or 1 for every $\alpha \in \Omega \setminus \gamma^*$, and let Ω_1 be the set of all α with $Ind_\gamma(\alpha) = 1$.*
For any $f \in H(\Omega)$, let N_f be the number of zeros in f in Ω_1, counted according to their multiplicites.

1. *If $f \in H(\Omega)$ and f has no zeros on γ^* then*

$$N_f = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = Ind_\Gamma(0)$$

where $\Gamma = f \circ \gamma$.

2. *If also $g \in H(\Omega)$ and*

$$|f(z) - g(z)| < |f(z)|, \qquad \text{for all } z \in \gamma^*$$

then $N_g = N_f$.

Note: *Part (2) is called* Roche's Theorem. *It says that two holomorphic functions have the same number of zeros in Ω_1, if they are close together on the boundary of Ω_1, as specified by the inequality above.*

Proof. Put $\varphi = f'/f$, a meromorphic function in Ω. If $a \in \Omega$ and f has a zero of order $m = m(a)$ at a, then $f(z) = (z-a)^m h(z)$, where h and $1/h$ are holomorphic in some neighborhood V of a. In $V \setminus \{a\}$,

$$\varphi = \frac{f'}{f} = \frac{m}{z-a} + \frac{h'}{h}.$$

Thus

$$\mathrm{Res}(\varphi; a) = m(a).$$

Let $A = \{a \in \Omega_1 : f(a) = 0\}$. If our assumptions about the index of γ are combined with the residue theorem one obtains

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{a\in A}\mathrm{Res}(\varphi; m) = \sum_{a\in A} m(a) = N_f.$$

This proves the first half of the proof of (1). The other half is a matter of direct computation.

$$\begin{aligned}\mathrm{Ind}_\Gamma(0) &= \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{d}z}{z} = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\Gamma'(s)}{\Gamma(s)}\,\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f'(\gamma(s))}{f(\gamma(s))}\gamma'(s)\,\mathrm{d}s = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z\,.\end{aligned}$$

The parameter interval of γ was here taken to be $[0, 2\pi]$.
Next, Rouche's Theorem shows that g has no zeros on γ^*. Hence, we can apply (1) with g in place of f. Put $\Gamma_0 = g \circ \gamma$. Then it follows from (1), (2), and the homotopy equality that

$$N_g = \mathrm{Ind}_{\Gamma_0}(0) = \mathrm{Ind}_\Gamma(0) = N_f.$$

□

This same statement is presented mathematically equivalently, but also slightly different notation, so for completeness, we will present the alternative statement of Rouche's Theorem.

Theorem 2.12 (Rouche's Theorem [3])**.** *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z)| > |g(z)| \qquad \text{for all } z \in C,$$

then f and $f+g$ have the same number of zeros inside the circle C.

Theorem 2.13 (Open Mapping Theorem [3])**.** *If f is holomorphic and nonconstant in a region Ω, then f is open.*

Proof. Let w_0 belong to the image of f, say $w_0 = f(z_0)$. We most prove that all points w near w_0 also belong to the image of f.
Define $g(z) = f(z) - w$ and write

$$\begin{aligned} g(z) &= (f(z) - w_0) + (w_0 - w) \\ &= F(z) + G(z). \end{aligned}$$

Now choose $\delta > 0$ such that the disc $|z - z_0| \leq \delta$ is contained in Ω, and $f(z) = \omega_0$ on the circle $|z - z_0| = \delta$. We then select $\epsilon > 0$ so that we have $|f(z) - w_0| \geq \epsilon$ on the circle $|z - z_0| = \delta$. Now if $|w - w_0| < \epsilon$, we have $|F(z)| > |G(z)|$ on the circle $|z - z_0| = \delta$, and by Rouche's theorem, we conclude that $g = F + G$ has a zero inside the circle since F has one. □

Theorem 2.14 (Maximum Modulus Theorem [3])**.** *If f is a nonconstant holomorphic function on a region Ω, then f cannot attain a maximum in Ω.*

Proof. Suppose that f did attain a maximum at z_0. Since f is holomorphic it is an open mapping, and therefore, if $D \subset \Omega$ is a small disc centered at z_0, its image $f(D)$ is open and contains $f(z_0)$. This proves that there are points $z \in D$ such that $|f(z)| > |f(z_0)|$, a contradiction. □

Theorem 2.15 (Schwarz Lemma [2])**.** *Suppose $f \in H^\infty$ (the space of all bounded holomorphic functions in U), $\|f\|_\infty \leq 1$, and $f(0) = 0$. Then*

$$|f(z)| \leq |z| \qquad (z \in U) \tag{1}$$

$$|f'(0)| \leq 1; \tag{2}$$

if equality holds in 1 for one $z \in U \setminus \{0\}$, or if equality holds in 2, then $f(z) = \lambda z$, where λ is a constant, $\lambda| = 1$, which is a rotation.

Remark 2.3 (Geometric Applications of Schwarz Lemma[2])**.** The hypothesis is that f is a holomorphic mapping of U into U which keeps the origin fixed; part of the conclusion is that either f is a rotation or f moves each $z \in U \setminus \{0\}$ closer to the origin that it was.

Proof. Since $f(0) = 0$, then $f(z)/z$ has a removable singularity at $z = 0$. Hence there exists $g \in H(U)$ such that $f(z) = zg(z)$. If $z \in U$ and $|z| < r < 1$. then

$$|g(z)| \leq \max_\theta \frac{|f(re^{i\theta})|}{r} \leq \frac{1}{r}.$$

Letting $r \to 1$, we see that $|g(z)| \leq 1$ at every $z \in U$. This gives (1). Since $f'(0) = g(0)$, (2) follows. If $|g(z)| = 1$ for some $z \in U$, then g is constant, by another application of the maximum modulus principle. □

2.3 More Technical Theorems

Theorem 2.16 (Weierstrauss factorization [2])**.** *Let f be an entire function, supposed $f(0) \neq 0$, and let $z_1, z_2, z_3, \ldots$ be the zeros of f, listed according to their multiplicities. Then there exists an entire function g and a sequence $\{p_n\}$ of nonnegative integers, such that*

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right),$$

where $E_0(z) = 1 - z$, and for $p = 1, 2, 3, \ldots$

$$E_p(z) = (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)$$

are Weierstrauss's elementary factors. ***Note****: If f has a zero of order k at $z = 0$, the preceding applies to $f(z)/z^k$. The factorization is not unique; a unique factorization can be associated with those f whose zeros satisfy the condition required for the convergence of a canonical product.*

Remark 2.4 (Comparing Weierstrauss and Hadamard [3])**.** The following factorization of Hadamard refined this result by showing that in the case of functions of finite order, the degree of the canonical factors can be taken to be constant, and g is then a polynomial.

Theorem 2.17 (Hadamard Factorization [3])**.** *Supposed f is entire and has growth order ρ_0. Let k be the integer so that $k \leq \rho_0 < k + 1$. If $a_1, a_2, \ldots$ denote the (non-zero) zeros of f, then*

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right),$$

where P is a polynomial of degree $\leq k$, and m is the order of the zero of f at $z = 0$.

Theorem 2.18 (Riemann Mapping Theorem [2])**.** *Every simply connected region Ω in the plane (other than the plane itself) is conformally equivalent to the open unit disc U.* ***Note****: The case of the plane clearly has to be excluded, by Liouville's theorem. Thus the plane is not conformally equivalent to U, although the two regions are homeomorphic. The only property of simply connected regions which will be used in the proof is that every holomorphic function which has no zero in such a region has a holomorphic square root there.*

2.4 Techniques

3 Real Analysis

3.1 Background Material

Definition 3.1 (Dense[1])**.** A set $E \subseteq X$ is *dense* in X if $\overline{E} = X$.

Definition 3.2 (Limsup and Liminf of Sets[1])**.** If $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of subsets of $\mathbb{R}^d$, then we define

$$\limsup_{k\to\infty} E_k = \bigcap_{j=1}^{\infty}\left(\bigcup_{k=j}^{\infty} E_k\right) \quad \text{and} \quad \liminf_{k\to\infty} E_k = \bigcup_{j=1}^{\infty}\left(\bigcap_{k=j}^{\infty} E_k\right).$$

Definition 3.3 (G_δ-sets and F_σ-set [1])**.**

1. A set $H \subseteq \mathbb{R}^d$ is a *G_δ-set* if there exist countably many open sets U_k sith that $H = \cap U_k$.

2. A set $H \subseteq \mathbb{R}^d$ is a *F_σ-set* if there exist countably many closed sets F_k such that $H = \cup F_k$.

More tersely, [2] defines G_δ-sets and F_σ-sets as "all countable intersections of open sets" and "all countable unions of closed sets", respectively.

Remark 3.1. Both [1, 2] explain the historical rationale for these names, which is included to help solidify the difference for studying. The notation is due to Hausdorff. The letters F and G were used for closed and open sets, respectively, since F comes from the French *fermé*(closed) and G comes from the German *Gebiet* (neighborhood, open set). The σ refers to union (*Summe*), δ to intersection (*Durchschnitt*).

Example 3.1 (Clopen sets are both G_δ-sets and F_σ-sets [1])**.** The half open interval $[a, b)$ is neither an open nor a closed subset of $\mathbb{R}$, but it is both a G_δ-set and an F_σ-set because we can write it as

$$\bigcap_{k=1}^{\infty} \left(a - \frac{1}{k}, b\right) = [a, b) = \bigcup_{k=1}^{\infty} \left[a, b - \frac{1}{k}\right]$$

Definition 3.4 (Topology [1])**.** The *topology*, τ of X, is the set of all open subsets of X.

Using logic of set theory, we can easily show that $\emptyset \in \tau$, as the null set is open. Additionally, $X \subseteq X$, so it must be open, i.e. $X \in \tau$. Lastly, we know that a <u>finite</u> intersection of open sets is open, and the union of infinitely many open sets is still open. In other words, a topology is closed under unions, finite intersections, contains the whole set, the null set, and every open set in between. This is further solidified in Rudin's definition.

Definition 3.5 (Topology [2])**.** A collection τ of subsets of a set X is said to be a *topology in* X if τ has the following three properties:

1. $\emptyset \in \tau$ and $X \in \tau$.

2. If $V_i \in \tau$ for $i = 1, \cdots, n$, then $V_1 \cap V_2 \cap \cdots \cap V_n \in \tau$.

3. If $\{V_\alpha\}$ is an arbitrary collection of members of τ (finite, countable, or uncountable), then $\cup_\alpha V_\alpha \in \tau$.

Definition 3.6 (Topological Space [2])**.** If τ is a topology in X, then X is called a *topological space*, and members of τ are called *open sets* in X.

Definition 3.7 (Continuous [2])**.** Finally, if X and Y are topological spaces and if f is a mapping of X into Y, then f is said to be *continuous* provided that $f^{-1}(V)$ is an open set in X for every open set V in Y.

Definition 3.8 (Vector Space)**.**

3.2 Key Definitions

Definition 3.9 (Outer Lebesgue Measure [1]). The *exterior Lebesgue measure* (or the *outer Lebesgue measure*) of a set $E \subseteq \mathbb{R}^d$ is

$$|E|_e := \inf \left\{ \sum_k \operatorname{vol}(Q_k) \right\},$$

where the infimum is taken over all countable collections of boxes $\{Q_k\}$ such that $E \subseteq \cup Q_k$.

Definition 3.10 (Lebesgue Measure [1]). A set $E \subseteq \mathbb{R}^d$ is *Lebesgue measurable*, or simply *measurable* for short, if

$$\forall \epsilon > 0, \exists \text{ open } U \supseteq E \text{ such that } |U \setminus E|_e \leq \epsilon$$

From this definition it is easy to define an alternative definition of Lebesgue measure that instead of open sets that contain the set of interest, we have closed sets that are contained in the set of interest.

Lemma 3.1 (Alternative Defintion of Lebesgue Measure[1]). *A set $E \subset \mathbb{R}^d$ is* Lebesgue measurable *if and only if for each $\epsilon > 0$, there exists a closed set $F \subset E$ such that $|E \setminus F|_e < \epsilon$.*

Proof. E is measurable iff $E^C = \mathbb{R}^d \setminus E$ is measurable iff there exists and open set $U \supset E^C$ such that $|U \setminus E^C| < \epsilon$ iff $F = U^C$, F is closed and satisfies $E \setminus F = U \setminus E^C$ iff $|E \setminus F| < \epsilon$. □

Definition 3.11 (σ-algebra [1]). Let X be a set, and let Σ be a family of subsets of X (in other words, $\Sigma \subseteq \mathcal{P}(X)$, the power set of X). If

1. Σ is nonempty,
2. Σ is closed under complements,
3. Σ is closed under countable unions,

then Σ is called a *σ-algebra* of subsets of X.

Rudin similarly defines σ-algebras as:

Definition 3.12 (σ-algebra [2]). A collection $\mathfrak{M}$ of subsets of a set X is said to be a *σ-algebra in X* if $\mathfrak{M}$ has the following properties:

1. $X \in \mathfrak{M}$
2. If $A \in \mathfrak{M}$, then $A^C \in \mathfrak{M}$, where A^C is the complement of A relative to X.
3. If $A = \cup_{n=1}^{\infty} A_n$ and if $A_n \in \mathfrak{M}$ for $n = 1, 2, 3, \ldots$, then $A \in \mathfrak{M}$.

Then [2] immediately uses this definition to define measurable sets, measurable spaces, and measurable functions; something [1] does at the end to summarize all of the types of sets that he proved are measurable.

Definition 3.13 (Measurable Space, Measurable Sets [2]). If $\mathfrak{M}$ is a σ-algebra in X, then X is called a *measurable space*, and the members of $\mathfrak{M}$ are called the *measurable sets* in X.

Definition 3.14 (Measurable Functions [2]). If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

Heil similarly defines a Lebesgue measurable function as such:

Definition 3.15 (Extended Real-Valued Measurable Functions [1]). Let $E \subseteq \mathbb{R}^d$ and $f : E \to [-\infty, \infty]$ be given. We say that f is *Lebesgue measurable function* on E, or simply a *measuralbe function* for short, if

$$\{f > a\} = f^{-1}(a, \infty])$$

is a measurable subset of $\mathbb{R}^d$ for each number $a \in \mathbb{R}$.

Rudin has additional comments to the last definitions, which states $\emptyset = X^C$, so since $X \in \mathfrak{M}$ then $X^C = \emptyset \in \mathfrak{M}$. Also we can set $A_{n+1} = A_{n+2} = \cdots = \emptyset$, so given $A_i \in \mathfrak{M}$ for $i = 1, ..., n$, then $\cup_{i=1}^n A_i \in \mathfrak{M}$. Additionally, since

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^C \right)^C,$$

$\mathfrak{M}$ is closed under under the formation of countable (and also finite) intersections. Lastly, assuming $A, B \in \mathfrak{M}$, then $A - B := A \setminus B := A \cap B^C \in \mathfrak{M}$.

Remark 3.2 (Putting the σ in σ-algebra [2, 4]). The prefix σ refers to the fact that in 3.11 and 3.12, if we restrict it to finite unions only (which then allows us to include finite intersections), then we have what is called an *algebra*, which is still closed under complements and finite unions. This means that algebras are a proper subset of σ-algebras.

A less terse and more systematic discussion about measurable sets is provided by Heil.

Notation 3.1 (Measurable set [1]). The collection of all Lebesgue measurable subsets of $\mathbb{R}^d$ will be denoted by

$$\mathcal{L} = \mathcal{L}(\mathbb{R}^d) = \{E \subseteq \mathbb{R}^d : E \text{ is Lebesgue measurable}\},$$

but this doesn't tell us what measurable sets can look like.

Lemma 3.2 (Open Sets are Measurable [1]). *If $U \subseteq \mathbb{R}^d$ is open, then U is Lebesgue measurable, and therefore $U \in \mathcal{L}$.*

Lemma 3.3 (Null Sets are Measurable [1]). *If $Z \subseteq \mathbb{R}^d$ and $|Z|_e = |Z| = 0$, then*

Theorem 3.4 (Closure Under Countable Unions [1]). *If $E_1, E_2, \cdots$ are all measurable subsets of $\mathbb{R}^d$, then their union $E = \cup E_k$ is also measureable, and*

$$|E| \leq \sum_{k=1}^{\infty} |E_k|$$

Corollary 3.4.1 (Boxes are Measurable [1]). *Every box, Q in $\mathbb{R}^d$ is a Lebesgue measurable set as it can be decomposed into the union of the interior Q°, which is open, and the boundary, ∂Q, which has measure zero.*

Theorem 3.5 (Compact Sets are Measurable [1]). *Every compact subset of* $\mathbb{R}^d$ *is Lebesgue measurable.*

Corollary 3.5.1 (Closed Sets are Measurable [1]). *Every closed subset of* $\mathbb{R}^d$ *is Lebesgue measurable. This is because we can write every closed set,* E *as a countable union of compact sets,* F_k*.*

Theorem 3.6 (Closure under Complements [1]). *If* $E \subseteq \mathbb{R}^d$ *is Lebesgue measurable, then so is* $E^C := \mathbb{R}^d \setminus E$*.*

Corollary 3.6.1 (Closure under Countable Intersections [1]). *If the sets* $E_1, E_2, \dots \subseteq \mathbb{R}^d$ *are each Lebesgue measurable, then so is* $E = \cap E_k$*. This is because* $\cup E_k^C = \cap (E_k)^C$*.*

Corollary 3.6.2 (Closure under Relative Complements [1]). *If* A *and* B *are Lebesgue measurable subsets of* $\mathbb{R}^d$*, then so is* $A \setminus B := A \cap B^C$*.*

Definition 3.16 (Borel Set [1]). Let $\mathcal{U} = \{U \subseteq \mathbb{R}^d : U \text{ is open}\}$. be the collection of all open subsets of $\mathbb{R}^d$, or a topology of $\mathbb{R}^d$. Let $\mathcal{B} = \Sigma(\mathcal{U})$ be the σ-algebra generated by $\mathcal{U}$. The elements of $\mathcal{B}$ are called *Borel subsets* of $\mathbb{R}^d$, and $\mathcal{B}$ is the *Borel* σ*-algebra* on $\mathbb{R}^d$.

Corollary 3.6.3 (Borel Measurable Sets [1]).

1. $\mathcal{B}$ *contains every open set, closed set,* G_δ*-set,* F_σ*-set,* $G_{\delta\sigma}$*-set,* $F_{\sigma\delta}$*-set, and so forth.*
2. $\mathcal{B} \subseteq \mathcal{L}$*, or every element of* $\mathcal{B}$ *is a Lebesgue measurable set.*
3. *And if* $E \subseteq \mathbb{R}^d$ *is Lebesgue measurable, then* $E = B \setminus Z$ *where* $B \in \mathcal{B}, |Z| = 0$*, or every Lebesgue measurable set differs from a Borel set by at most a set of measure zero.*
4. *There do exist Lebesgue measurable sets that are not Borel sets such as* $g^{-1}(N)$*, where* $g(x) = \phi(x) + x, \phi(x)$ *is the Cantor-Lebesgue function, and* N *is the constructed nonmeasurable set.*

Measure is probably the most important concept of integration theory, so it is super important that we understand this in the concrete sense of Lebesgue or Borel Measure, but also in the abstract sense. The following definitions will help us.

Definition 3.17 (Measure Space, Measure [4]). A *measure space* consists of a set X equipped with two fundamental objects:

1. a σ-algebra $\mathcal{M}$ of "measurable" sets, which is a non-empty collection of subsets of X closed under complements and countable unions and intersections.
2. A *measure* $\mu : \mathcal{M} \to [0, \infty]$ with the following defining property: if $E_1, E_2, ...$ is a countable family of disjoint sets in $\mathcal{M}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

Similarly, [2] defines positive measure as

Definition 3.18 (Positive Measure [2]). A *positive measure* is a function μ, defined on a σ-aglebra $\mathfrak{M}$, whose range is in $[0, \infty]$ and which is countably additive. This means that if $\{A_i\}$ is a disjoint countable collection of members of $\mathfrak{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

To avoid trivalities, we shall also assume that $\mu(A) < \infty$ for at least one $A \in \mathfrak{M}$.

Definition 3.19 (Measure Space [2]). A *measure space* is a measurable space which has a positive measure defined on the σ-algebra of its measureable sets.

Definition 3.20 (Complex Measure [2]). A *complex measure* is a complex-valued countably additive function defined on a σ-algebra.

Now that we have defined what an abstract measure is, we should next discuss properties:

Theorem 3.7 (Properties of a Positive Measure [2]). *Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then*

1. $\mu(\emptyset) = 0$
2. $\mu(A_1 \cup A_2 \cup \cdots \cup A_n) = \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n)$ *(finite additivity)*
3. $A \subseteq B$ *imples* $\mu(A) \leq \mu(B)$ *if* $A \in \mathfrak{M}, B \in \mathfrak{M}$. *(monotonicity)*
4. $\mu(A_n) \to \mu(A)$ *as* $n \to \infty$ *if* $A = \cup_{n=1}^{\infty} A_n, A_n \in \mathfrak{M}$, *and*

   $$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots. \qquad \textit{(continuity from below)}$$

5. $\mu(A_n) \to \mu(A)$ *as* $n \to \infty$ *if* $A = \cap_{n=1}^{\infty} A_n, A_n \in \mathfrak{M}$, *and*

   $$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \qquad \textit{(continuity from above)}$$

   and $\mu(A_1)$ *is finite.*

The last two concepts are presented slightly differently in [1] because it included monotonicity in there as well. For completeness, it is also presented here

Theorem 3.8 (Continuity from Below[1]). *If $E_1, E_2, ...$ are measurable subsets of $\mathbb{R}^d$ such that $E_1 \subseteq E_2 \subseteq \cdots$, then $|E_1| \leq |E_2| \leq \cdots$ and*

$$\left|\bigcup_{k=1}^{\infty} E_k\right| = \lim_{k\to\infty} |E_k|.$$

Theorem 3.9 (Continuity from Above[1]). *If $E_1 \supseteq E_2 \supseteq \cdots$ are measurable subsets of $\mathbb{R}^d$ and $|E_k| < \infty$ for some k, then $|E_1| \geq |E_2| \geq \cdots$ and*

$$\left|\bigcap_{k=1}^{\infty} E_k\right| = \lim_{k\to\infty} |E_k|.$$

The continuity from above presented in [1] becomes the one presented in [2] if you define $\min_{k\in\mathbb{N}} E_k : |E_k| < \infty := A_1$.

Theorem 3.10 (Cartesian Products [1]). *If $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ are Lebesgue measurable sets, then $E \times F \subseteq \mathbb{R}^{m+n}$ is a Lebesgue measurable subset of $\mathbb{R}^{m+n}$, and*

$$|E \times F| = |E||F|.$$

Now that we have defined properties of an abstract measure, let's define the properties of an exterior measure.

Definition 3.21 (Exterior Measure [4]). Let X be a set. An *exterior measure* (or *outer measure*) μ_* on X is a function μ_* from the collection of all subsets of X to $[0, \infty]$ that satisfies the following properties:

1. $\mu_*(\emptyset) = 0$
2. If $E_1 \subseteq E_2$, then $\mu_*(E_1) \leq \mu_*(E_2)$.
3. If $E_1, E_2, ...$ is a countable family of sets, then

$$\mu_* \left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} \mu_*(E_j) \qquad \text{(countable subadditivity)}$$

Now that we have an idea of measurable sets in hand, we need to look at measurable functions to solidify the notion of integration theory. The Riemann integral is based off of the class of step functions, with each give a finite sum.

Definition 3.22 (Step Function [4]). A *step function* is given as a finite sum

$$f = \sum_{k=1}^{N} c_k \chi_{R_k},$$

where each R_k is a rectangle, and the c_k are constants.

However, we need something slightly more general for the stronger notion of a Lebesgue integral; this is where the simple function comes in.

Definition 3.23 (Simple Function, Standard Representations[1]). Let $E \subseteq \mathbb{R}^d$ be a Lebesgue measurable set. A *simple function* on E is a measurable function $\phi : E \to \mathbb{C}$ that takes only finitely many distinct values. The *standard representation* of a simple function ϕ is the representation, $\phi = \sum_{k=1}^{N} c_k \chi_{E_k}$ where $c_1, ..., c_N$ are the distinct values taken by ϕ and $E_k = \phi^{-1}(c_k) = \{\phi = c_k\}$ for $k = 1, ..., N$.

Example 3.2. $\phi = \chi_{[0,2]} + \chi_{[1,3]}$ is a simple function on $\mathbb{R}$ because it takes only three distinct values. Its standard representation is

$$\phi = 0\chi_{E_1} + 1\chi_{E_2} + 2\chi_{E_3},$$

where $E_1 = (-\infty, 0) \cup (3, \infty), E_2 = [0, 1) \cup (2, 3]$, and $E_3 = [1, 2]$.Of course we could also write ϕ in the form

$$\phi = 1\chi_{E_2} + 2\chi_{E_3},$$

but while the sets E_2, E_3 are disjoint, they do not partition the domain $\mathbb{R}$. In general, one of the scalars c_k in the standard representation of a simple function ϕ might be zero.

Definition 3.24 (Really Simple Function [1])**.** A *really simple function* on $\mathbb{R}$ is a measurable function ϕ of the form

$$\phi = \sum_{k=1}^{N} c_k \chi_{[a_k, b_k)},$$

where $N \in \mathbb{N}, a_k < b_k$ are real numbers, and c_k is a scalar.

Now that we have a notion of measurable sets and measurable functions, we marry those two concepts in integration theory with the following definition.

Definition 3.25 (L^1-norm and Integrable Functions [1])**.** Let $E \subseteq \mathbb{R}^d$ be a measurable set, and let $f : E \to \bar{\mathbf{F}}$ be a measurable function on E.

1. The extended real number
$$\|f\|_1 = \int_E |f|$$
is called the *L^1-norm of f on E* (it could be infinite).

2. We say that f is *integrable* on E if $\|f\|_1 = \int_E |f| < \infty$.

Remark 3.3. Although we refer to $\|\cdot\|_1$ as a "norm," it is actually only a seminorm on the space of integrable functions because $\|f\|_1 = 0$ if and only if $f = 0$ a.e.

Now that we have defined what measurable functions and integrable functions are, a natural next step is to talk about classes or specaes of these measurable functions to study what sort of shared proerties they all have.

Definition 3.26 (The Lebesgue space $L^\infty(E)$[1])**.** If E is a measurable subset of $\mathbb{R}^d$, then the *Lebesgue space of essentially bounded functions on E* is the set of all essentially bounded measurable functions $f : E \to \bar{\mathbf{F}}$. That is,

$$L^\infty(E) = \left\{ f : E \to \bar{\mathbf{F}} : f \text{ is measurable and } \|f\|_\infty < \infty \right\}.$$

where the *L^∞-norm* (actually a seminorm) is defined as

$$\|f\|_\infty = \operatorname{esssup}_{x \in E} |f(x)|.$$

Remark 3.4. Recall that the *uniform norm* of a function f on E is

$$\|f\|_u = \sup_{x \in E} |f(x)|.$$

If f is continuous function whose domain is an open set $U \subseteq \mathbb{R}^d$, then $\|f\|_\infty = \|f\|_u$. However, in general we only have the inequality $\|f\|_\infty \le \|f\|_u$.

Definition 3.27 (The Lebesgue Space $L^2(E)$[4])**.** Let E be a measurable subset of $\mathbb{R}^d$. Then the *Lebesgue space of L^2 functions* on E is the equivalence class of measurable functions for which $\int_E |f(x)^2| \, \mathrm{d}\mu(x) < \infty$. The norm is then

$$\|f\|_{L^2(E,\mu)} = \left(\int_E |f(x)|^2 \, \mathrm{d}\mu(x) \right)^{1/2}.$$

There is also an inner product on this space given by

$$(f, g) = \int_E f(x) \overline{g(x)} \, \mathrm{d}\mu(x).$$

Finally, as opposed to defining every $p \in \mathbb{R}$ as its own Lebesgue space of L^p functions, we are going to give a more generic example (all the special cases are listed above though.)

Definition 3.28 (The Lebesgue Space $L^p(E)$[1])**.** Let E be a measurable subset of $\mathbb{R}^d$.

1. If $0 < p < \infty$ and $f : E \to \bar{\mathbf{F}}$ is measurable, then we say that f is p-integrable if $\int_E |f|^p < \infty$. In that case we set,

$$\|f\|_p = \left(\int_E |f|^p \right)^{1/p}.$$

If f is not p-integrable then we take $\|f\|_p = \infty$. We define $L^p(E)$ to be the set of all p-integrable functions on E, and call $L^p(E)$ the *Lebesgue space of p-integrable functions on E.*

2. If $p = \infty$, then $L^\infty(E)$ is the set of all measurable functions $f : E \to \bar{\mathbf{F}}$ that are essentially bounded. That is, f belongs to $L^\infty(E)$ if

$$\|f\|_\infty = \operatorname{esssup}_{x \in E} |f(x)| < \infty.$$

We call $L^\infty(E)$ the *Lebesgue space of essentially bounded functions on E.*

Remark 3.5 (L^p spaces are vector spaces, for $1 \leq p \leq \infty$)**.** It is not hard to prove that L^p spaces are linear spaces, i.e. closed under vector addition and scalar multiplication. Similarly, it is not hard to show that L^p spaces are naturally equipped with a seminorm that has nonnegativity, homogeneity, the triangle inequality, and almost everywhere uniqueness. The standard argument works for $1 \leq p < \infty$ and can separately be shown for L^∞ recalling the definition of an essential supremum.

Definition 3.29 (Smooth Function[4])**.** A function that is indefinitely differentiable are referred to as *smooth functions*, or C^∞ functions.

Definition 3.30 (Compactly Supported Functions[1])**.** The *support* of a continuous function f on $\mathbb{R}^d$ is the closure in $\mathbb{R}^d$ of the set of points where f is nonzero:

$$\operatorname{supp}(f) = \overline{\{x \in \mathbb{R}^d : f(x) \neq 0\}}.$$

We say that f has *compact support* if supp(f) is a compact set. Lastly, we say that the space of continuous functions with compact support is

$$C_c(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) : \text{ supp}(f) \text{ is compact}\}.$$

Definition 3.31 (Schwartz Space $\mathcal{S}$, Schwartz functions [1])**.** The *Schwartz space*, $\mathcal{S}$, or the *space of rapidly decreasing functions on* $\mathbb{R}$ is

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m D^n f \in L^\infty \text{ for all } m, n \geq 0\}$$

where $D^k = f^{(k)} = \frac{\mathrm{d}^k}{\mathrm{d}x^k} f$ is a differential operator. Also a function in the Schwartz space is sometimes called a *Schwartz function.*

Definition 3.32 (Singular Function [1])**.** A function f on $[a, b]$ (either extended real-value or complex valued) is *singular* if f is differentiable at almost every point in $[a, b]$ and $f' = 0$ a.e. on $[a, b]$.

Definition 3.33 (Bounded Variation [1])**.** Let $f : [a,b] \to \mathbb{C}$ be given. For each finite partition

$$\Gamma = \{a = x_0 < \cdots < x_n = b\}$$

of $[a,b]$, set

$$S_\Gamma = S_\Gamma[f; a, b] = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| .$$

The *total variation* of f over $[a,b]$(or simply the *variation* of f, for short) is

$$V[f] = V[f; a, b] = \sup\{S_\Gamma : \Gamma \text{ is a partition of } \; [a,b]\}.$$

We say that f has *bounded variaton* on $[a,b]$ if $V[f; a, b] < \infty$. We collect the functions that have bounded variation on $[a,b]$ to form the space

$$\mathrm{BV}[a,b] = \{f : [a,b] \to \mathbb{C} : f \text{ has bounded variation}\}.$$

Definition 3.34 (Absolutely Continuous Functions [1])**.** We say that a function $f : [a,b] \to \mathbb{C}$ is *absolutely continuous on* $[a,b]$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any finite or countably infinite collection of nonoverlapping subintervals $\{[a_j, b_j]\}$ of $[a,b]$, we have

$$\sum_j (b_j - a_j) < \delta \quad \implies \quad \sum_j |f(b_j) - f(a_j)| < \epsilon.$$

We denote the class of absolutely continuous functions on $[a,b]$ by

$$\mathrm{AC}[a,b] = \{f : [a,b] \to \mathbb{C} : f \text{ is absolutely continuous on } [a,b]\}.$$

So far all of the measures that we have discussed have had the property that $0 \leq \mu(E) \leq \infty$, but in fact there are measures that can be positive or negative, which is what we defined as a "signed measure".

Definition 3.35 (Signed Measure [4])**.** A *signed measure* possesses all the properties of a measure, except that it may take positive or negative values. More precisely, a *signed measure* ν on a σ-algebra $\mathcal{M}$ is a mapping that satisfies:

1. The set function ν is extended-valued in the sense that $-\infty < \nu(E) \leq \infty$ for all $E \in \mathcal{M}$.

2. If $\{E_j\}_{j=1}^{\infty}$ are disjoint subsets of $\mathcal{M}$, then

$$\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu(E_j).$$

Definition 3.36 (Signed Measure [2])**.** Let's consider a real measure μ on a σ-algebra $\mathfrak{M}$. (Such measures are frequently called *signed measures*.) Define the total variation $|\mu|$ as $\mu(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|$ where the supremum are being taken over all partitions $\{E_i\}$ of E. Now define

$$\mu^+ = \frac{1}{2}(|\mu| + \mu), \qquad \mu^- = \frac{1}{2}(|\mu| - \mu).$$

Then both μ^+ and μ^- are positive measures on $\mathfrak{M}$, and they are bounded. Also

$$\mu = \mu^+ - \mu^-, \qquad |\mu| = \mu^+ + \mu^-.$$

The measures μ^+ and μ^- are called *positive* and *negative variations* of μ, respectively. This representation of μ as the difference of positive measures μ^+ and μ^- is known as the *Jordan decomposition* of μ.

Definition 3.37 (Absolutely Continuous [2]). Let μ be a positive measure on a σ-algebra $\mathfrak{M}$, and let λ be an arbitrary measure on $\mathfrak{M}$; λ may be positive or complex. We say that λ is *absolutely continuous* with respect to μ, and write

$$\lambda \ll \mu$$

if $\lambda(E) = 0$ for every $E \in \mathfrak{M}$ for which $\mu(E) = 0$

Definition 3.38 (Concentrated [2]). If there is a set $A \in \mathfrak{M}$ such that $\lambda(E) = \lambda(A \cap E)$ for every $E \in \mathfrak{M}$, we say that λ is *concentrated on* A. This is equivalent to the hypothesis that $\lambda(E) = 0$ whenever $E \cap A = \emptyset$.

Definition 3.39 (Mutually Singular [2]). Supposed λ_1 and λ_2 are measures on $\mathfrak{M}$, and supposed there exists a pair of disjoint sets A and B such that λ_1 is concentrated on A and λ_2 is concentrated on B. Then we say that λ_1 and λ_2 are *mutually singular*, and write

$$\lambda_1 \perp \lambda_2$$

.

3.3 Core Results

Theorem 3.11 (Countable Subadditivity[1]). *If $E_1, E_2, \ldots$ are countably many sets in $\mathbb{R}^d$, then*

$$\left|\bigcup_{k=1}^{\infty} E_k\right|_e \leq \sum_{k=1}^{\infty} |E_k|_e .$$

Proof. If any particular E_k has infinite exterior measure then both sides of the equation are ∞, so we are dine in this case. Therefore, assume that $|E_k|_e < \infty$ for every k, and fix $\epsilon > 0$. We know that for each k we can find a covering $\left\{Q_j^{(k)}\right\}_j$ of E_k by countably many boxes such that

$$\sum_j \operatorname{vol}\left(Q_j^{(k)}\right) \leq |E_k|_e + \frac{\epsilon}{2^k}.$$

Then $\left\{Q_j^{(k)}\right\}_{j,k}$ is a covering of $\cup_k E_k$ by countably many boxes, so

$$\begin{aligned}\left|\bigcup_{k=1}^{\infty} E_k\right|_e &\leq \sum_{k=1}^{\infty}\sum_j \operatorname{vol}\left(Q_j^{(k)}\right) \\ &\leq \sum_{k=1}^{\infty}\left(|E_k|_e + \frac{\epsilon}{2^k}\right) \\ &= \left(\sum_{k=1}^{\infty}|E_k|_e\right) + \epsilon\end{aligned}$$

Since ϵ is arbitrary, the result follows. $\square$

Theorem 3.12 (Countable Additivity [1]). *If $E_1, E_2, \ldots$ are disjoint Lebesgue measurable subsets of $\mathbb{R}^d$, then*

$$\left|\bigcup_{k=1}^{\infty} E_k\right| = \sum_{k=1}^{\infty} |E_k| \,.$$

Proof. Step 1. Assume first that each set E_k is bounded. From subadditivity we obtain

$$\left|\bigcup_{k=1}^{\infty} E_k\right|_e \leq \sum_{k=1}^{\infty} |E_k|_e \,,$$

so our task is to prove the opposite inequality.
Fix $\epsilon > 0$. By the alternative definiton of a measurable set, there exists a closed set $F_k \subset E_k$ such that

$$|E_k \setminus F_k| < \frac{\epsilon}{2^k}.$$

Since E_k is bounded, F_k is compact. Hence $\{F_k\}_{k\in\mathbb{N}}$ is a collection of disjoint compact sets. Let N be any finite positive integer. Then, by using the fact that disjoint compact sets are countably additive and monotonicity, we see that

$$\sum_{k=1}^{N} |F_k| = \left|\bigcup_{k=1}^{N} F_k\right| \leq \left|\bigcup_{k=1}^{N} E_k\right| \leq \left|\bigcup_{k=1}^{\infty} E_k\right| .$$

Taking the limit as $N \to \infty$,

$$\sum_{k=1}^{\infty} |F_k| = \lim_{N\to\infty} \sum_{k=1}^{N} |F_k| \leq \left|\bigcup_{k=1}^{\infty} E_k\right| .$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{\infty} |E_k| &= \sum_{k=1}^{\infty} |F_k \cup (E_k \setminus F_k)| \\
&\leq \sum_{k=1}^{\infty} (|F_k| + |E_k \setminus F_k|) \qquad \text{(finite subadditivity)} \\
&\leq \sum_{k=1}^{\infty} \left(|F_k| + \frac{\epsilon}{2^k}\right) \qquad \text{(alt. definition of measure)} \\
&= \left(\sum_{k=1}^{\infty} |F_k|\right) + \epsilon \\
&\leq \left|\bigcup_{k=1}^{\infty} E_k\right| + \epsilon \qquad \text{(limit of measure of compact sets)}
\end{aligned}$$

Since ϵ is arbitrary, it follows.
Step 2. Now assume that $E_1, E_2, \ldots$ are arbitrary disjoint measurable subsets of $\mathbb{R}^d$. Set

$$E_k^j = \{x \in E_k : j - 1 \leq \|x\| < j\}, \qquad \text{for } j, k \in \mathbb{N}.$$

Then $\{E_k^j\}_{k,j}$ is a countable collection of disjoint **bounded** measurable sets. For each fixed $k \in \mathbb{N}$ we have

$$\bigcup_{j=1}^{\infty} E_k^j = E_k,$$

and furthermore
$$\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty} E_k^j = \bigcup_{k=1}^{\infty} E_k = E.$$
Therefore,
$$\begin{aligned}
\left|\bigcup_{k=1}^{\infty} E_k\right| &= \left|\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty} E_k^j\right| && \text{(by above)}\\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |E_k^j| && \text{(by Step 1)}\\
&= \left|\bigcup_{j=1}^{\infty} E_k^j\right| && \text{(by Step 1)}\\
&= \sum_{k=1}^{\infty} |E_k| && \text{(by above).}
\end{aligned}$$

□

Definition 3.40 (Algebra, premeasure [4])**.** Let X be a set. An *algebra* in X is a non-empty collection of subsets of X that is closed under complements, finite unions, and finite intersections. Let $\mathcal{A}$ be an algebra in X. A *premeasure* on an algebra $\mathcal{A}$ is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ that satisfies:

1. $\mu_0(\emptyset) = 0$.
2. If $E_1, E_2, ...$ is a countable collection of disjoint sets in $\mathcal{A}$ with $\cup_{k=1}^{\infty} E_k \in \mathcal{A}$, then
$$\mu_0\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu_0(E_k).$$

In particular, μ_0 is finitely additive on $\mathcal{A}$.

Remark 3.6 (Premeasure and Outer Measure[4])**.** The example of the Lebesgue outer measure belongs to a large class of exterior measures that can all be obtained using "coverings" by a family of special sets whose measures are taken as known. This idea is systematized by the notion of a "premeasure", and premeasures give rise to exterior measures in a natrual way. In fact, we can extend the premeasure to a measure if we have a σ-algebra generated by our algebra on which the presemausre is defined.

Theorem 3.13 (Carathéodory's Separation Condition [4])**.** *A set E in X is* Carathéodory measurable*, or simply* measurable*, if one has*
$$\mu_*(A) = \mu_*(E \cap A) + \mu_*(E^C \cap A), \quad \textit{for every } A \subset X.$$
Moreover, given an exterior measure μ_* on a set X, the collection $\mathcal{M}$ of Carathédory measurable sets form a σ-algebra. Morover μ_* restricted to $\mathcal{M}$ is a measure.

Proof. A first observation we make is that to prove a set E is measurable, it suffices to verify
$$\mu_*(A) \geq \mu_*(E \cap A) + \mu(E^C \cap A)$$

since the reverse inequality is automatically verified by the subadditivity property of exterior measure. Clearly, $\emptyset$ and X belong to $\mathcal{M}$ and the symmetry inherent in conditon 3.3 shows that $E^C \in \mathcal{M}$ whenever $E \in \mathcal{M}$. Thus $\mathcal{M}$ is nonempty and closed under complements. Next, we prove that that $\mathcal{M}$ is closed under finite unions of disjoint sets, and μ_* is finitely additive on $\mathcal{M}$. Indeed if $E_1, E_2 \in \mathcal{M}$, and A is any subset of X, then

$$\begin{aligned}
\mu_*(A) &= \mu_*(E_2 \cap A) + \mu_*(E_2^C \cap A) \\
&= \mu_*(E_1 \cap E_2 \cap A) + \mu_*(E_1^C \cap E_2 \cap A) + \mu_*(E_1 \cap E_2^C \cap A) + \mu_*(E_1^C \cap E_2^C \cap A) \\
&\geq \mu_*((E_1 \cup E_2) \cap A) + \mu_*((E_1 \cup E_2)^C \cap A)
\end{aligned}$$

where in the first two lines we have used the measurability condition on E_2 and then E_1, and where the last inequality was obtained using the subadditivity of μ_* and the fact that $E_1 \cup E_2 = (E_1 \cap E_2) \cup (E_1^C \cap E_2) \cup (E_1 \cup E_2^C)$. Therefore, we have that $E_1 \cup E_2 \in \mathcal{M}$, and if E_1 and E_2 are disjoint, we find

$$\begin{aligned}
\mu_*(E_1 \cup E_2) &= \mu_*(E_1 \cap (E_1 \cup E_2)) + \mu_*(E_1^C \cap (E_1 \cup E_2)) \\
&= \mu_*(E_1) + \mu_*(E_2)
\end{aligned}$$

Finally, it suffies to show that $\mathcal{M}$ is closed under countable unions of disjoint sets, and that μ_* is countably additive on $\mathcal{M}$. Let $E_1, E_2, ...$ denote a countable collection of disjoint sets in $\mathcal{M}$, and define

$$G_n = \bigcup_{j=1}^{n} E_j \quad \text{and} \quad G = \bigcup_{j=1}^{\infty} E_j.$$

For each n, the set G_n is a finite union of sets in $\mathcal{M}$, and hence $G_n \in \mathcal{M}$. Moreover, for any $A \subset X$ we have

$$\begin{aligned}
\mu_*(G_n \cap A) &= \mu_*(E_n \cap (G_n \cap A)) + \mu_*(E_n^C \cap (G_n \cap A)) \\
&= \mu_*(E_n \cap A)) + \mu_*(G_{n-1} \cap A) \\
&= \sum_{j=1}^{n} \mu_*(E_j \cap A),
\end{aligned}$$

where the last equality is obtained by induction. Since we know that $G_n \in \mathcal{M}$, and $G^C \subset G_n^C$, we find that

$$\mu_*(A) = \mu_*(G_n \cap A) + \mu_*(G_n^C \cap A) \geq \sum_{j=1}^{n} \mu_*(E_j \cap A) + \mu_*(G^C \cap A)$$

Letting n tend to infinity, we obtain

$$\begin{aligned}
\mu_*(A) \geq \sum_{j=1}^{n} \mu_*(E_j \cap A) + \mu_*(G^C \cap A) &\geq \mu_*(G \cap A) + \mu_*(G^C \cap A) \\
&= \mu_*(A).
\end{aligned}$$

Therefore all of the inequalities above are equalities, and we conclude that $G \in \mathcal{M}$, as desired. Moreover, taking $A = G$ in the above, we find that μ_* is countably additive on $\mathcal{M}$, and the proof of the theorem is complete. $\square$

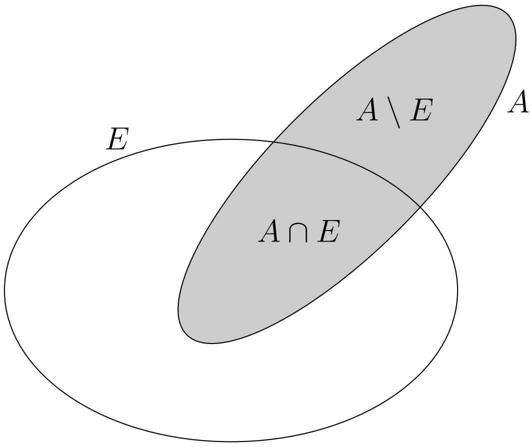

Figure 1: If E is measurable, then $|A \cap E|_e$ and $|A \setminus E|_e$ must sum to $|A|_e$ for *every* set A.

Theorem 3.14 (Carathéodory's Criterion [1]). *A set $E \subset \mathbb{R}^d$ is Lebesgue measurable if and only if*

$$\forall A \subseteq \mathbb{R}^d, \quad |A|_e = |A \cap E|_e + |A \setminus E|_e.$$

Proof. $\Rightarrow$. Suppose that E is measurable, and fix any set $A \subseteq \mathbb{R}^d$. Since $A = (A \cap E) \cup (A \setminus E)$, subadditivity implies that

$$|A|_e \leq |A \cap E|_e + |A \setminus E|_e. \tag{3}$$

There exists a G_δ-set $H \supseteq A$ such that $|H| = |A|_e$. We can write H as a disjoint union $H = (H \cap E) \cup (H \setminus E)$. Since Lebesgue measure is countably additive on measurable sets and since H and E are measurable, we conclude that

$$\begin{aligned} |A|_e = |H| &= |H \cap E| + |H \setminus E| && \text{(countable additivity)} \\ &\geq |A \cap E|_e + |A \setminus E|_e && \text{(monotonicity)} \end{aligned}$$

$\Leftarrow$. Let E be any subset of $\mathbb{R}^d$ that satisfies equation 3. For each $k \in \mathbb{N}$, let $E_k = E \cap B_k(0)$. Fix $\epsilon > 0$, and let U be an open set that contains E_k and satisfies

$$|E_k|_e \leq |U| \leq |E_k|_e + \epsilon \quad \text{(regularity)}.$$

By replacing U with $U \cap B_k(0)$ if necessary, we can assume that $U \subseteq B_k(0)$. Using equation 3, we compute that

$$\begin{aligned} |E_k|_e + |U \setminus E_k|_e &= |U \cap E_k|_e + |U \setminus E_k|_e && \text{(since } E_k \subseteq U) \\ &= |U \cap E|_e + |U \setminus E|_e && \text{(since } U \subseteq B_k(0)) \\ &= |U| && \text{(by equation 3)} \\ &\leq |E_k|_e + \epsilon \end{aligned}$$

Since $|E_k|_e$ is finite, we can subtract it from both sides to obtain that $|U \setminus E_k|_e \leq \epsilon$. Thus E_k is measurable, and therefore $E = \cup E_k$ is measurable as well. □

Lemma 3.15 (Borel - Cantelli). *Suppose that sets $E_k \subseteq \mathbb{R}^d$ satisfy $\sum |E_k|_e < \infty$. Then* $\liminf E_k$ *and* $\limsup E_k$ *each have exterior zero.*

Proof. Recall that the definition of limsup is

$$\limsup_{k\to\infty} E_k = \bigcap_{j=1}^{\infty}\left(\bigcup_{k=j}^{\infty} E_k\right).$$

This means that we can use the following properties, we can bound the measure of the limit superior.

$$\begin{aligned}
\left|\limsup_{k\to\infty} E_k\right|_e &= \left|\bigcap_{j=1}^{\infty}\left(\bigcup_{k=j}^{\infty} E_k\right)\right|_e \\
&\leq \left|\bigcup_{k=j}^{\infty} E_k\right|_e \\
&\leq \sum_{k=j}^{\infty} |E_k|_e \\
&< \infty
\end{aligned}$$

where the first inequality is due to monotonicity of the fact that $\cap(\cup E_k) \subseteq \cup E_k$, and the second is the finite subadditivity of exterior measure. Lastly, we use the fact that $\sum |E_k|_e < \infty$ by hypothesis.
This last fact can only happen if it converges to something and a convergent series means that the tail of that convergent series must approach zero, or

$$\lim_{k\to\infty} \sum_{k=j}^{\infty} |E_k|_e = 0$$

Lastly, putting all the pieces together and since it is an exterior measure (and not a signed measure) we know that

$$0 \leq \left|\limsup_{k\to\infty} E_k\right|_e \leq 0$$

which implies that $\left|\limsup_{k\to\infty} E_k\right|_e = 0$.
The same argument can be used for the liminf recalling that it is defined as $\liminf_{k\to\infty} E_k = \cup_{j=1}^{\infty}\left(\cap_{k=j}^{\infty} E_k\right)$. This of course means that we have to apply subadditivity and then monotonicity.
This also extends to measure if we further assume that $\{E_k\}_{k=1}^{\infty}$ are all measurable. □

Remark 3.7 (Littlewood's three principles[4])**.** Although the notions of measurable sets and measurable functions represent new tools, we should not overlook their relation to the older concepts they replaced. Littlewood aptly summarized these connections in the form of three principles that provide a useful intuitive guide in the initial study of the theory.

1. Every set is nearly a finite union of intervals.

2. Every function is nearly continuous. (Luzin's theorem)

3. Every convergent sequence is nearly uniformly convergent. (Egoroff's theorem)

Theorem 3.16 (Measure of finite cubes[4]). *Suppose E is a measurable subset of $\mathbb{R}^d$. Then for every $\epsilon > 0$, if $m(E)$ is finite, there exists a finite union $F = \cup_{j=1}^{N} Q_j$ of closed cubes such that*

$$m(E \triangle F) \leq \epsilon,$$

*where $E \triangle F$ is the **symmetric difference** between the sets E and F, defined by $E \triangle F = (E \setminus F) \cup (F \setminus E)$, which consists of those points that belong to only one of the two sets, E or F.*

Proof. Choose a family of closed cubes $\{Q_j\}_{j=1}^{\infty}$ so that

$$E \subset \bigcup_{j=1}^{\infty} Q_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} |Q_j| \leq m(E) + \frac{\epsilon}{2}.$$

Since $m(E) < \infty$, the series converges and there exists $N > 0$ such that $\sum_{j=N+1}^{\infty} |Q_j| < \epsilon/2$. If $F = \cup_{j=1}^{N} Q_j$, then

$$\begin{aligned} m(E \triangle F) &= m(E \setminus F) + m(F \setminus E) \\ &\leq m\left(\bigcup_{j=N+1}^{\infty} Q_j\right) + m\left(\bigcup_{j=1}^{\infty} Q_j - E\right) \\ &\leq \sum_{j=N+1}^{\infty} |Q_j| + \sum_{j=1}^{\infty} |Q_j| - m(E) \\ &\leq \epsilon. \end{aligned}$$

□

Theorem 3.17 (Luzin's Theorem[1]). *Let E be a bounded, measurable subset of $\mathbb{R}^d$, and let $f : E \to \bar{\boldsymbol{F}}$ be measurable and finite a.e. Then for each $\epsilon > 0$, there exists a closed set $F \subset E$ such that $|E \setminus F| < \epsilon$ and $f\big|_F$ is continuous.*

Proof. Step 1. Let $\phi = \sum_{k=1}^{N} c_k \chi_{E_k}$ be the standard representation of a simple function ϕ on E, and fix $\epsilon > 0$. Since each subset E_k is measurable, the alternative definition of measure says that there exists closed sets $F_k \subseteq E_k$ such that

$$|E_k \setminus F_k| < \frac{\epsilon}{N}, \qquad \text{for } k = 1, ..., N.$$

The set $F = F_1 \cup \cdots \cup F_N$ is closed, and since $E_1, ..., E_N$ partition E. we have $|E \setminus F| < \epsilon$. Since E is bounded, the sets $F_1, ..., F_N$ are compact and disjoint. Consequently, F_j is separated from F_k by a positive distance when $j \neq k$. Since ϕ is constant on each individual set F_k, it follows that the restriction of ϕ to F is continuous.

Step 2. Now let f be an arbitrary measurable function on E, and fix $\epsilon > 0$. We know that there exist simple functions ϕ_n that converge pointwise to f on E. Applying Step 1, for each integer $n > 0$ we can find a closed set $F_n \subseteq E$ such that

$$|E \setminus F_n| < \frac{\epsilon}{2^{n+1}} \qquad \text{and} \qquad \phi_n\bigg|_{F_n} \text{ is continuous.}$$

By Egorov's Theorem, there exists a measurable set $A \subseteq E$ with measure $|A| < \epsilon/4$ such that ϕ_n converges to f uniformly on $E \setminus A$. Also by the alternative definition of measure, there exists a closed set $F_0 \subseteq E \setminus A$ such that

$$|(E \setminus A) \setminus F_0| < \frac{\epsilon}{4}.$$

Writing $E \setminus F_0 = (E \setminus A) \setminus F_0 \cup A$, we see that

$$|E \setminus F_0| \leq |(E \setminus A) \setminus F_0| + |A| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

Further, ϕ_n converges to f uniformly on F_0 since F_0 is contianed in $E \setminus A$.
Next, let

$$F = \bigcap_{n=0}^{\infty} F_n.$$

Since F is closed and bounded, it is compact. Further,

$$|E \setminus F| = \left| \bigcup_{n=0}^{\infty} (E \setminus F_n) \right| \leq \sum_{n=0}^{\infty} |E \setminus F_n| < \sum_{n=0}^{\infty} \frac{\epsilon}{2^{n+1}} = \epsilon.$$

Since ϕ_n is continuous on F_n, it is continuous on the smaller set F. Thus $\{\phi_n\big|_F\}_{n\in\mathbb{N}}$ is a sequence of continuous functions that converge uniformly on F to $f\big|_F$. Therefore $f\big|_F$ is continuous because the uniform limit of a sequence of continuous functions is continuous. □

Theorem 3.18 (Egorov's Theorem [1]). *Let E be a measurable subset of $\mathbb{R}^d$ with $|E| < \infty$. Suppose that $\{f_n\}$ is a sequence of measurable funcitons on E (either complex-valued or extended real-valued) sich that $f_n \to f$ a.e., where f is finite a.e.. Then for each $\epsilon > 0$ there exists a measurable set $A \subset E$ such that*

1. $|A| < \epsilon$
2. *f_n converges uniformly to f on $E \setminus A$, i.e.,*

$$\lim_{n\to\infty} \|(f - f_n) \cdot \chi_{A^C}\|_u = \lim_{n\to\infty} \left(\sum_{x \notin A} |f(x) - f_n(x)| \right) = 0.$$

Proof. Case 1: Complex-Valued Functions. Assume that the f_n are complex valued. Since the pointwise a.e. limit of measurable functions is measurable, we know that f is measurable.
Let Z be the set of points where $f_n(x)$ does not converge to $f(x)$. By hypothesis, Z has measure zero. In order to quantify more precisely the points where $f_n(x)$ is far from $f(x)$, for each $k \in \mathbb{N}$ we let

$$Z_k = \left\{ x \in E : |f(x) - f_n(x)| \geq \frac{1}{k} \text{ for infinitely many } n \right\}.$$

Since $Z_n \subset Z$ we have $|Z_n| = 0$. By the Borel- Cantelli Lemma,

$$Z_k = \limsup_{n\to\infty} \left\{ |f - f_n| \geq \frac{1}{k} \right\} = \bigcap_{n=1}^{\infty} A_n(k),$$

where for $k, n \in \mathbb{N}$ we take

$$A_n(k) = \bigcup \left\{ |f - f_n| \geq \frac{1}{k} \right\}.$$

Each set $A_n(k)$ is measurable. By construction,

$$A_1(k) \supset A_2(k) \supset \qquad \text{and} \bigcap_{n=1}^{\infty} A_n(k) = Z_k.$$

Since $|E|$ has finite measure, we can therefore apply continuity from above to obtain

$$\lim_{n\to\infty} |A_n(k)| = |Z_k| = 0.$$

Fix any $\epsilon > 0$. For each integer $k \in \mathbb{N}$ we can find an integer $n_k \in \mathbb{N}$ such that

$$|A_n(k)| < \frac{\epsilon}{2^k}.$$

By subadditivity, the set

$$A = \bigcup_{k=1}^{\infty} A_{n_k}(k)$$

has measure $|A| < \epsilon$. Moreoever, if $x \notin A$ then $x \notin A_{n_k}(k)$ for any k, so $|f(x)-f_m(x)| < \frac{1}{k}$ for all $m \geq n_k$.
In summary, we have found a set A with measure $|A| < \epsilon$ such that for each integer k there exists an integer n_k such that

$$m \geq n_k \implies \sup_{x\notin A} |f(x) - f_m(x)| \leq \frac{1}{k}.$$

This says that f_n converges uniformly to f on $E \setminus A$.
Case 2: Extended Real-Valued Functions Now assume that f_n and f are extended real-valued functions with f finite a.e.. Let $Y = \{f = \pm\infty\}$ be the set of measure zero consisting of all points where $f(x) = \pm\infty$. Then $F = E \setminus Y$ is measurable, f is finite on F, and $f_n \to f$ a.e. on F. Now repeat the proof of Case 1 with E replaced by F. Although $f_n(x)$ van be $\pm\infty$, if $x \in F$ then $f(x) - f_n(x)$ never takes an indeterminate form, and the proof preceeds just as before to construct measurable set $A \subset F$ such that $|A| < \epsilon$ and $f_n \to f$ uniformly on $F \setminus A$. Consequently $B = A \cup Y$ is a measurable subset of E that satisfies $|B| = |A| < \epsilon$, and $f_n \to f$ uniformly on $E \setminus B$. □

Definition 3.41 (Integral of a Nonnegative Simple Function[1])**.** Let ϕ be a nonnegative simple function on a measurable set $E \subset \mathbb{R}^d$, and let $\phi = \sum_{k=1}^{N} c_k\chi_{E_k}$ be its standard representation. The *Lebesgue integral of ϕ over E* is

$$\int_E \phi = \int_E \phi(x)\,\mathrm{d}x = \sum_{k=1}^{N} c_k|E_k|.$$

Lemma 3.19 (Linearity and monotonicity of nonnegative simple functions [1])**.** *If ϕ and ψ are nonnegative simple functions defined on a measurable set $E \subset \mathbb{R}^d$ and $c \geq 0$, then the following statements hold.*

1. $\int_E (\phi + \psi) = \int_E \phi + \int_E \psi$ *and* $\int_E c\phi = c\int_E \phi$.
2. *If $E_1, ..., E_N$ are any measurable subsets of E and $c_1, ..., c_N$ are any nonnegative scalars, then*
$$\int_E \sum_{k=1}^{N} c_k\chi_{E_k} = \sum_{k=1}^{N} c_k|E_k|.$$

Proof. Trivial. They follow from the properties of nonnegative and simple functions □

Recall that we are able to approximate any nonnegative function by nonnegative simple funcitons.

Theorem 3.20 (Approximating nonnnegative functions by nonnegative simple functions[1])**.** *Let $E \subset \mathbb{R}^d$ be a measurable set, and let $f : E \to [0, \infty]$ be a nonnegative, measurable function on E.*

1. *There exist nonnegative simple functions ϕ_n such that $\phi_n \nearrow f$. That is $0 \le \phi_1 \le \phi_2 \le \cdots$, and $\lim_{n\to\infty} \phi_n(x) = f(x)$ for each $x \in E$.*
2. *If f is bounded on some set $A \subseteq E$, then we can construct the functions ϕ_n in statement (a) so that they converge uniformly to f on A, i.e.,*

$$\lim_{n\to\infty} \|(f - \phi_n) \cdot \chi_A\|_u = \lim_{n\to\infty} \left(\sup_{x\in A} |f(x) - \phi_n(x)| \right) = 0.$$

Proof. Not Trivial, but not relevant. □

Definition 3.42 (Lebesgue Integral of a Nonnegative Function[1])**.** Let $E \subset \mathbb{R}^d$ be a measurable set. If $f : E \to [0, \infty]$ is a measurable function, then the *Lebesgue integral of f over E* is

$$\int_E f = \int_E f(x)\,\mathrm{d}x = \sup \left\{ \int_E \phi : 0 \le \phi \le f, \phi \text{ simple} \right\}.$$

Definition 3.43 (Positive and Negative Parts[1])**.** Given an extended real-valued function $f : X \to [-\infty, \infty]$, the *positive part* of f is

$$f^+(x) = \max\{f(x), 0\},$$

and the *negative part* of f is

$$f^-(x) = \max\{-f(x), 0\}.$$

By construction, f^+ and f^- are nonnegative extended real-valued functions, and we have

$$f = f^+ - f^- \qquad \text{and} \qquad |f| = f^+ + f^-.$$

Definition 3.44 (Lebesgue Integral of an Extended Real-Valued Function[1])**.** Let $f : E \to [-\infty, \infty]$be a measurable extended real-valued function defined on a measurable set $E \subset \mathbb{R}^d$. The *Lebesgue integral of f over E* is

$$\int_E f = \int_E f^+ - \int_E f^-,$$

as long as this does not have the form $\infty - \infty$ (in that case, the integral is undefined).

Definition 3.45 (Lebesgue Integral of a Complex-Valued Function[1])**.** Let $f : E \to \mathbb{C}$ be a measurable complex-valued function defined on a measurable set $E \subseteq \mathbb{R}^d$. Write f in real and imaginary parts as $f = f_r + if_i$, where f_r and f_i are real-valued. If $\int_E f_r$ and $\int_E f_i$ both exist and are finite, then the *Lebesgue integral of f over E* is

$$\int_E f = \int_E f_r + i \int_E f_i.$$

Otherwise, the integral is undefined.

Theorem 3.21 (Monotone Convergence Theorem [1]). *Let $E \subseteq \mathbb{R}^d$ be a measurable set, and let $f_n : E \to [0, \infty]$ be measurable functions on E such that $f_n \nearrow f$. Then*

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. By hypothesis, for each $x \in E$ we have (in the extended real sense) that

$$f_1(x) \le f_2(x) \le \cdots \qquad \text{and} \qquad f(x) = \lim_{n\to\infty} f_n(x).$$

Consequently, since f_n are all nonnegative, we at least have the inequalities

$$0 \le \int_E f_1 \le \int_E f_2 \le \cdots \le \int_E f \le \infty.$$

Note that we have not assumed that any of the integrals in the preceding line are finite. However, an increasing sequence of nonnegative extended real numbers must converge to a nonnegative extended real number, so

$$I = \lim_{n\to\infty} \int_E f_n$$

exists in the extended real sense. Further, it follows from equation 3.3 that $0 \le I \le \int_E f \le \infty$. We must prove that $I \ge \int_E f$.
Let ϕ be any simple function such that $0 \le \phi \le f$, and fix $0 < \alpha < 1$. Set $E_n = \{f_n \ge \alpha\phi\}$, and observe that

$$E_1 \subseteq E_2 \subseteq \cdots .$$

Further, $\cup E_n = E$ (this is where we use the assumption $\alpha < 1$). The continuity from below property of the integral implies that $\int_{E_n} \phi \to \int_E \phi$. Consequently,

$$\begin{aligned}
I &= \lim_{n\to\infty} \int_E f_n && \text{(definition of } I\text{)} \\
&= \limsup_{n\to\infty} \int_E f_n && \text{(lim = limsup when limit exists)} \\
&\ge \limsup_{n\to\infty} \int_{E_n} f_n && \text{(since } E_n \subseteq E\text{)} \\
&\ge \limsup_{n\to\infty} \int_{E_n} \alpha\phi && \text{(by definition of } E_n\text{)} \\
&= \alpha \int_E \phi && \text{(by linearity and continuity from below)}
\end{aligned}$$

Letting $\alpha \to 1$, we see that $I \ge \int_E \phi$. Finally, by taking the supremum over all such simple functions ϕ we obtain the inequality $I \ge \int_E f$. □

Lemma 3.22 (Fatou's Lemma [1]). *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative measurable functions on a measurable set $E \subseteq \mathbb{R}^d$, then*

$$\int_E \left(\liminf_{n\to\infty} f_n\right) \le \liminf_{n\to\infty} \int_E f_n.$$

In particular, if $f_n(x) \to f(x)$ for each $x \in E$, then

$$\int_E f \le \liminf_{n\to\infty} \int_E f_n.$$

Proof. Define
$$f(x) = \liminf_{n\to\infty} f_n(x) = \lim_{k\to\infty} \inf_{n\geq k} f_n(x) = \lim_{k\to\infty} g_k(x)$$
where
$$g_k(x) = \inf_{n\geq k} f_n(x).$$
The functions g_k increase monotonically to f, i.e., $g_k \nearrow f$. The Monotone Convergence Theorem therefore implies that
$$\int_E f = \lim_{k\to\infty} \int_E g_k.$$
However, $g_k \leq f_k$ and therefore $\int g_k \leq \int f_k$ for every k. Consequently,
$$\int_E f = \lim_{k\to\infty} \int_E g_k = \liminf_{k\to\infty} \int_E g_k \leq \liminf_{k\to\infty} \int_E f_k.$$
The second more particular equation follows by recalling that if the limit of a sequence exists, then it equals the liminf of the sequence. □

Theorem 3.23 (Dominated Convergence Theorem [1])**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions (either extended real-valued or complex-valued) defined on a measurable set $E \subset \mathbb{R}^d$. If*

1. *$f(x) = \lim_{n\to\infty} f_n(x)$ exists for a.e. $x \in E$, and*
2. *there exists a single integrable function g such that for each $n \in \mathbb{N}$ we have $|f_n(x)| \leq g(x)$ a.e.,*

then f_n converges to f in L^1-norm, i.e.
$$\lim_{n\to\infty} \|f - f_n\|_1 = \lim_{n\to\infty} \int_E |f - f_n| = 0.$$
As a consequence,
$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. The hypotheses imply that g is integrable and nonnegative almost everywhere. Therefore
$$0 \leq \int_E g = \int_E |g| < \infty$$
Step 1. Suppose first that $f_n \geq 0$ a.e. for each n. In that case we can apply Fatou's Lemma to obtain
$$0 \leq \int_E f = \int_E \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_E f_n \leq \int_E g < \infty.$$

We also have $g - f_n \geq 0$ a.e., so we can apply Fatou's Lemma to the functions $g - f_n$. Doing this, we obtain

$$\begin{aligned}
\int_E g - \int_E f = \int_E (g-f) & \qquad (f \text{ and } g \text{ are integrable}) \\
= \int_E \liminf_{n\to\infty} (g - f_n) & \quad (\text{since } f_n \to f \text{ a.e.}) \\
\leq \liminf_{n\to\infty} \int_E (g - f_n) & \quad (\text{Fatou's Lemma}) \\
= \liminf_{n\to\infty} \left(\int_E g - \int_E f_n \right) & \quad (f_n \text{ and } g \text{ are integrable}) \\
= \int_E g - \limsup_{n\to\infty} \int_E f_n & \quad (\text{properties of liminf}).
\end{aligned}$$

All of the integrals that appear in the preceding calculation are finite, so by rearranging we see that $\limsup_{n\to\infty} \int_E f_n \leq \int_E f$. Combining this with what we have shown by properties of f_n, f. yields

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n \leq \limsup_{n\to\infty} \int_E f_n \leq \int_E f.$$

Hence $\lim_{n\to\infty} \int_E f_n$ exists and equals $\int_E f$. This does not show that f_n converges to f in L^1-norm, but we will establish that in Step 2.

Step 2. Now assume that the f_n are arbitrary functions (either extended real-valued or complex-valued) that satisfy the hypotheses. In this case, the functions $|f - f_n|$ are nonnegative a.e., converges pointwise a.e. to the zero function, and satisfy

$$|f - f_n| \leq |f| + |f_n| \leq 2g \text{ a.e.}$$

Since $2g$ is integrable, we can apply Step 1 to $|f - f_n|$, which gives us

$$\lim_{n\to\infty} \|f - f_n\|_1 = \lim_{n\to\infty} \int_E |f - f_n| = \int_E 0 = 0.$$

This proves that f_n converges to f in L^1-norm. Applying that f, f_n are both integrable and converge in the L^1-norm, we also get in the limit case $\lim_{n\to\infty} \int_E f_n = \int_E f$. $\square$

Corollary 3.23.1 (Bounded Convergence Theorem [1])**.** *Let E be a measurable subset of $\mathbb{R}^d$ such that $|E| < \infty$. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions on E such that $f_n \to f$ a.e. and there exists a single finite constanct M such that $|f_n| \leq M$ a.e. for every n, then $f_n \to f$ in L^1-norm.*

Proof. Since $|E| < \infty$, the constant function M is integrable. The result therefore follows by applying the DCT with $g(x) = M$. $\square$

Theorem 3.24 (Riesz-Fischer Theorem[4])**.** *The vector space L^1 is complete in its metric.*

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in the norm, so that $\|f_n - f_m\| \to 0$ as $n, m \to \infty$. The plan of the proof is to extract a subsequence of $\{f_n\}$ that converges to f, both pointwise almost everywhere and in the norm.

Under ideal circumstances we would have that the sequence $\{f_n\}$ converges almost everywhere to a limit f, and we would then prove that the sequence converges to f also in the

		pointwise a.e. convergence		L^p-norm convergence		
		$\Downarrow$ if $\lvert E\rvert < \infty$		$\Downarrow$		
L^∞-norm convergence	$\Longrightarrow$	almost uniform convergence	$\Longrightarrow$	convergence in measure	$\Longrightarrow$	pointwise a.e. convergence of a subsequence
		$\Downarrow$				
		pointwise a.e. convergence				

Figure 2: Convergence of function spaces and their implications

norm. Unfortunately, almost everywhere convergence does not hold for general Cauchy sequences. The main point, however is that if the convergence in the norm is rapid enough, then almost everywhere convergence is a consequence, and this can be achieved by dealing with an appropriate subsequence of the original sequence.
Indeed, consider a subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}$ with the following property:

$$\left\|f_{n_{k+1}} - f_{n_k}\right\| \leq 2^{-k}, \qquad \forall k \geq 1.$$

The existence of such a subsequence is guaranteed by the fact that $\|f_n - f_m\| \leq \epsilon$ whenever $n, m \geq N(\epsilon)$, so it sufficies to take $n_k = N(2^{-k})$.
We now consider the series whose convergence will be seen below,

$$f(x) = f_{n_1}(x) + \sum_{k=1}^\infty \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$$

and

$$g(x) = |f_{n_1}(x)| + \sum_{k=1}^\infty \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|,$$

and note that

$$\int |f_{n_1}| + \sum_{k=1}^\infty \int \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right| \leq \int |f_{n_1}| + \sum_{k=1}^\infty 2^{-k} < \infty.$$

So the monotone convergence theorem implies that g is integrable, and since $|f| \leq g$, hence so is f. In particular, the series defining f converges almost everywhere (by the construction of the telescoping series), we find that

$$f_{n_k}(x) \to f(x) \qquad \text{a.e. } x.$$

To prove that $f_{n_k} \to f$ in L^1 as well, we simply observe that $|f - f_{n_k}| \leq g$ for all k, and apply the dominated convergence theorem to get $\|f_{n_k} - f\|_{L^1} \to 0$ as k tends to infinity.

Finally, the last step of the proof consists in recalling that $\{f_n\}$ is Cauchy. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $\|f_n - f_m\|_{L^1} < \epsilon/2$. If n_k is chosen so that $n_k > N$, and $\|f_{n_k} - f\| < \epsilon/2$, then the triangle inequality implies

$$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \epsilon/2 + \epsilon/2 = \epsilon$$

whenever $n > N$. Thus $\{f_n\}$ has a limit f in L^1. □

However, there is nothing remarkably special about L^1 for it to be complete. In fact, [1] states something much stronger! Recall, that being a Banach space is a complete normed vector space.

Theorem 3.25 ($L^p(E)$ is a Banach Space[1])**.** *Let E be a measurable subset of $\mathbb{R}^d$ and fix $1 \leq p \leq \infty$. If we identify functions that are equal almost everywhere, then $\|\cdot\|_p$ is a norm on $L^p(E)$ and $L^p(E)$ is complete with respect to this norm.*

Sometimes you don't even know where to start if you want to show that a property holds for all functions in $L^p(E)$, but if you can show it works for an "easy" special subclass of funtions and then extend it to then entire space by approximating the functions by the easier subclass.
To be more specific than the original definition of density, we provide equivalent conditions of density specifically for L^p spaces.

Lemma 3.26 (Dense subsets of $L^p(E)$[1])**.** *Let $E \subseteq \mathbb{R}^d$ be measurable, and fix $1 \leq p \leq \infty$. If S is a subset of $L^p(E)$, then T.F.A.E.*

1. *S is dense in $L^p(E)$, i.e., the closure of S equals $L^p(E)$.*
2. *If f is any element of $L^p(E)$, then there exists functions $f_n \in S$ such that $f_n \to f$ in L^p-norm.*
3. *If f is any element of $L^p(E)$, then for each $\epsilon > 0$ there exists a function $g \in S$ such that $\|f - g\|_p < \epsilon$.*

Now that we have these definitions let's look at some families of functions that are dense in $L^1(\mathbb{R}^d)$.

Theorem 3.27 (Dense Function Families in $L^1(\mathbb{R}^d)$[4])**.** *The following families of functions are dense in $L^1(\mathbb{R}^d)$*

1. *The simple functions.*
2. *The step functions.*
3. *The continuous functions of compact support.*

Proof. Let f be an integrable function on $\mathbb{R}^d$. First, we may assume that f is real-valued, because we may approximate its real and imaginary party independently. If this is the case, we may then write $f = f^+ - f^-$, where $f^+, f^- \geq 0$, and it now suffices to prove the theorem when $f \geq 0$.
For the simple functions, we know that there exists an increasing sequence of nonnegative simple functions $\{\phi_k\}_{k=1}^{\infty}$ that converge pointwise to f. By the dominated convergence theorem (or even simply the monotone convergence theorem), we then have

$$\|f - \phi_k\|_{L^1} \to 0 \qquad \text{as } k \to \infty.$$

Thus there are simple functions that are arbitrary close to f in the L^1 norm.
For the step functions, we first note that by the density of the simple functions, it sufficies to approximate simple functions by step functions. Then, we recall that a simple functions is a fiite linear combination of characteristic functions of sets of finite measure, so it suffies to show that if E is such a set, then there is a step function ψ so that $\|\chi_E - \psi\|_{L^1}$ is small. However, we now recall that there is an almost disjoint family of rectangles $\{R_j\}$ with $m(E \triangle \cup_{j=1}^{M} R_j) \leq 2\epsilon$. This χ_E and $\psi = \sum_j \chi_{R_j}$ differ at most on a set of measure 2ϵ, and as a result we find that $\|\chi_E - \psi\|_{L^1} < 2\epsilon$.
By the density of step functions in L^1, it suffieces to establish the continuous functions of compact support when f is the characteristic function of a rectangle. In the one-dimensional case, where f is the characteristic functino of an interval $[a, b]$, we may choose a continuous piecewise linear function g defined by

$$g(x) = \begin{cases} 1, & \text{if } a \leq x \leq b, \\ 0, & \text{if } x \leq a - \epsilon \text{ or } x \geq b + \epsilon \end{cases}$$

and with g linear on the intervals $[a - \epsilon, a]$ and $[b, b + \epsilon]$. Then $\|f - g\|_{L^1} < 2\epsilon$. In d dimensions, it suffices to note that the characteristic functions of a rectangle is the product of characteristic functions of intervals. Then, the desired continuous function of compact support is simply the product of functions like g defined above.
The results above for $L^1(\mathbb{R}^d)$ lead immediately to an extension in which $\mathbb{R}^d$ can be replaced by any fixed subset E of positive measure. In fact in E is such a subset, we can define $L^1(E)$ and carry out the arguments that are analogous to $L^1(\mathbb{R}^d)$. Better yet, we can proceed by extending any function f on E by setting $\tilde{f} = f$ on E and $\tilde{f} = 0$ on E^C, and defining $\|f\|_{L^1(E)} = \left\|\tilde{f}\right\|_{L^1(\mathbb{R}^d)}$. □

Again, L^1 is not that special and the above arguments can be extended to L^p spaces, with some exceptions of L^∞-spaces.

Theorem 3.28 (Compactly Supported Functions are Dense[1])**.** *Let $E \subseteq \mathbb{R}^d$ be a measurable set. If $1 \leq p < \infty$, then*

$$L^p_c(E) = \{f \in L^p(E) : f \textit{ is compactly supported}\}$$

is dense in $L^p(E)$.

Proof. Choose $f \in L^p(E)$, and for each $N \in \mathbb{N}$ define $f_n := f \cdot \chi_{E \cap [-n,n]^d}$. Then $(f - f_n) \to 0$ pointwise a.e., and

$$|f - f_n|^p = |f \cdot \chi_{E \setminus [-n,n]^d}|^p \leq |f|^p \in L^p(E).$$

The Dominated Convergence Theorem therefore implies that $|f - f_n|^p \to 0$ in L^1-norm, which is precisely the same as saying that $f_n \to f$ in L^p-norm. Since each f_n is compactly supported, we conclude that the set of compactly supported functions in $L^p(E)$ is dense in $L^p(E)$. □

Remark 3.8 (Compactly Supported Functions aren't always dense in L^∞[1])**.** The conclusion of the above theorem can fail if $p = \infty$. For example, if $f = 1$ is the function that is identically 1, then $\|f - g\|_\infty \geq 1$ for every compactly supported function g. The constant function 1 cannot be well-approximated in L^∞-norm by compactly supported function.

Theorem 3.29 (Simple Functions are Dense.[1])**.** *Assume that $E \subseteq \mathbb{R}^d$ is measurable and fix $1 \leq p \leq \infty$. The set S of all simple functions in $L^p(E)$ is dense in $L^p(E)$. Additionally if p is finite, then the set S_c of all compactly supported simple functions on E is dense in $L^p(E)$.*

Theorem 3.30 (Continuous Functions are Dense.[1])**.** *The space $C_c(\mathbb{R}^d)$ consists of all continuous, compactly supported functions on $\mathbb{R}^d$. Then $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$. Also, with respect to the L^∞-norm, $C_c(\mathbb{R}^d)$ is dense in*

$$C_0(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) : \lim_{\|x\|\to\infty} f(x) = 0\},$$

where the limit means that for each $\epsilon > 0$ there exists some compact set K such that $|f(x)| < \epsilon$ for all $x \notin K$.

Theorem 3.31. *Fix $1 \leq p < \infty$. Let $\mathcal{R}$ be the set of all really simple functions on $\mathbb{R}$,*

$$\mathcal{R} = \left\{ \sum_{k=1}^{N} c_k \chi_{[a_k,b_k)} : N > 0, c_k \text{ scalar}, \ a_k < b_k \in \mathbb{R} \right\}$$

is dense in $L^p(\mathbb{R})$ when p is finite.

Theorem 3.32 (Orthonormal sets on Hilbert Spaces are Equivalent[4])**.** *The following properties of an orthonormal set $\{e_k\}_{k=1}^\infty$ are equivalent.*

1. *Finite linear combinations of elements in $\{e_k\}$ are dense in $\mathcal{H}$.*
2. *If $f \in \mathcal{H}$, and $(f, e_j) = 0$ for all j, then $f = 0$.*
3. *if $f \in \mathcal{H}$, and $S_N(f) = \sum_{k=1}^N a_k e_k$, where $a_k = (f, e_k)$, then $S_N(f) \to f$ as $N \to \infty$ in the norm.*
4. *If $a_k = (f, e_k)$, then $\|f\|^2 = \sum_{k=1}^\infty |a_k|^2$.*

Proof. $1 \implies 2$: Given $f \in \mathcal{H}$ with $(f, e_j) = 0$ for all j, we wish to prove that $f = 0$. By assumption, there exists a sequence $\{g_n\}$ of elements in $\mathcal{H}$ that are finite linear combinations of elements in $\{e_k\}$, and such that $\|f - g_n\|$ tends to 0 as n goes to infinity. Since $(f, e_j) = 0$ for all j, we must have $(f, g_n) = 0$ for all n; therefore an application of the Cauchy-Shwartz inequality gives

$$\|f\|^2 = (f, f) = (f, f - g_n) \leq \|f\|\|f - g_n\|, \quad \text{for all } n.$$

Letting $n \to \infty$ proves that $\|f\|^2 = 0$; hence $f = 0$.
$2 \implies 3$: For $f \in \mathcal{H}$, we define

$$S_N(f) = \sum_{k=1}^{N} a_k e_k, \qquad \text{where } a_k = (f, e_k),$$

and prove first that $S_N(f)$ converges to some element $g \in \mathcal{H}$. Indeed, one notices that the definitions of a_k inplies $(f - S_N(f)) \perp S_n(f)$, so the Pythagorean theorem gives us

$$\|f\|^2 = \|f - S_N(f)\|^2 + \|S_N(f)\|^2 = \|f - S_N(f)\| + \sum_{k=1}^{N} |a_k|^2.$$

Hence $\|f\|^2 \geq \sum_{k=1}^{N} |a_k|^2$, and letting N tend to infinity, we obtain **Bessel's inequality**

$$\sum_{k=1}^{\infty} |a_k|^2 \leq \|f\|^2$$

which implies that the series $\sum_{k=1}^{N} |a_k|^2$ converges. Therefore, $\{S_N(f)\}_{N=1}^{\infty}$ forms a Cauchy sequence in $\mathcal{H}$ since

$$\|S_N(f) - S_M(f)\|^2 = \sum_{k=M+1}^{N} |a)k|^2, \qquad \text{whenever } N > M.$$

Since $\mathcal{H}$ is complete, there exists a $g \in \mathcal{H}$ such that $S_N(f) \to g$ as N tends to infinity. Fix j, and noe that for all sufficiently large N, $(f - S_N(f), e_j) = a_j - a_j = 0$. Since $S_N(f)$ tends to g, we conclude that

$$(f - g, e_j) = o \qquad \text{for all } j.$$

Hence $f = g$ by assumption, and we have proved that $f = \sum_{k=1}^{\infty} a_k e_k$.
3 $\implies$ 4: Notice that we immediately get in the limit as N goes to infinity

$$\|f\|^2 = \sum_{k=1}^{\infty} |a_k|^2.$$

4 $\implies$ 1: We see from the proof of 2 that $\|f - S_N(f)\|$ converges to 0. Since each $S_N(f)$ is a finite linear combination of elements in $\{e_k\}$, we have completed the circle of implications. □

Remark 3.9 (When Bessel's and Parseval's inequalities work)**.** In particular, a closer look at the proof shows that Bessel's inequality holds for any orthonormal family $\{e_k\}$. In contrast, the identity

$$\|f\|^2 = \sum_{k=1}^{\infty} |a_k|^2, \qquad \text{where } a_k = (f, e_k),$$

Parseval's identity, holds if and only if $\{e_k\}_{k=1}^{\infty}$ is also an northnormal basis.

On this topic, [1] has a very similar statement and proof, but also has some slight differences which will be helpful.

Theorem 3.33 (Orthnormal sequence in a Hilbert Space[1])**.** *If $\mathcal{E} = \{e_n\}_{n\in\mathbb{N}}$ is an orthnormal sequence in a Hilbert Space H, then the following statements hold.*

1. *Bessel's inequality: $\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2$ for each $x \in H$*
2. *If the series $x = \sum_{n=1}^{\infty} c_n e_n$ converges, then $c_n = \langle x, e_n\rangle$ for each $n \in \mathbb{N}$.*
3. *$\sum_{n=1}^{\infty} c_n e_n$ converges $\iff$ $\sum_{n=1}^{\infty} |c_n|^2 < \infty$.*

Similarly, we have another theorem that goes along with this theme of Hilbert spaces and bases.

Theorem 3.34 (Orthonormal sequences in Hilbert Spaces [1]). *If H is a Hilbert space and $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence in H, then T.F.A.E.*

1. *$\{e_n\}_{n\in\mathbb{N}}$ is complete, i.e. $\overline{span}\{e_n\}_{n\in\mathbb{N}} = H$.*
2. *For each $x \in H$, there exists a unique sequence of scalars $(c_n)_{n\in\mathbb{N}}$ such that $x = \sum c_n e_n$.*
3. *Every $x \in H$ satisfies*
$$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n,$$
where this series converges in the norm of H.
4. *Plancherel's Equality holds:*
$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2, \qquad \text{for all } x \in H.$$
5. *Parseval's Equality holds:*
$$\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x, e_n\rangle\langle e_n, y\rangle \qquad \text{for all } x, y \in H.$$

Proof. This proof involves essentially the same ideas as above. □

Theorem 3.35 (Finite dimensional Hilbert spaces have an orthonormal basis[1]). *If H is a finite-dimensional Hilbert space then H contains an orthonormal basis $\{e_1, ..., e_d\}$, where $d = \dim(H)$ is the dimension of the vector space H.*

Proof. Since H is a d-dimensional vector space, it has a Hamel basis, i.e., there is a set $\mathcal{B} = \{x_1, ..., x_d\}$ that is both linearly indenpendent and spans H. We will define a recursive procedure that constructs orthogonal vectors $y_1, ..., y_d$ that span H.
First, set $y_1 = x_1$, and note that $x_1 \neq 0$ since $x_1, ..., x_d$ are linearly independent. Define
$$M_1 = \text{span}\{x_1\} = \text{span}\{y_1\}.$$
If $d = 1$ then $M_1 = H$ and we stop here. Otherwise M_1 is a proper subset of H, and $x_2 \notin M_1$ (because $\{x_1, ..., x_d\}$ is linearly independent). Let p_2 be the orthogonal projection of x_2 onto M_1. Then $y_2 = x_2 - p_2$ is orthogonal to x_1, and $y_2 \neq 0$ since $x_2 \notin M_1$. Therefore, we can define
$$M_2 = \text{span}\{x_1, x_2\} = \text{span}\{y_1, y_2\},$$
where the second equality follows from the fact that y_1, y_2 are linear combinations of x_1, x_2 and vice, versa. Continuing in this way, we obtain orthogonal vectors $y_1, ..., y_d$ that span H. Hence $\{y_1, ..., y_d\}$ is an orthogonal, but not necessarily orthonormal, basis for H. Setting $e_k = y_k/\|y_k\|$ therefore gives us an orthonormal basis $\{e_1, ..., e_d\}$ for H. □

Theorem 3.36 (Infinite dimensional separable Hilbert spaces have an orthonormal basis[1]). *If H is an infinite-dimensional separable Hilbert space, then H contains an orthonormal basis of the form $\{e_n\}_{n\in\mathbb{N}}$.*

Proof. Since H is separable, it contains a countable dense subset $\{z_n\}_{n\in\mathbb{N}}$. The span of $\{z_n\}_{n\in\mathbb{N}}$ is dense in H, but $\{z_n\}_{n\in\mathbb{N}}$ need not be linearly independent. However, we can extract a subsequence that is independent and has the same span. Simply let x_1 be the first z_n that is nonzero. Then let x_2 be the first z_n after x_1 that is not a multiple of x_1. Then let x_3 be the first z_n after x_2 that does not belong to span$\{x_1, x_2\}$, and so forth. In this way, we obtain an independent sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\operatorname{span}\{x_n\}_{n\in\mathbb{N}} = \operatorname{span}\{z_n\}_{n\in\mathbb{N}}$. This span is dense in H by hypothesis. Now we apply the **Gram-Schmidt** procedure utilized in the prior proof to the vectors $x_1, x_2, ...,$ but without stopping. This gives us orthonormal vectors $e_1, e_2, ...$ such that for every n we have

$$\operatorname{span}\{e_1, ..., e_n\} = \operatorname{span}\{x_1, ..., x_n\}.$$

Consequently, $\operatorname{span}\{e_n\}_{n\in\mathbb{N}}$ equals $\operatorname{span}\{x_n\}_{n\in\mathbb{N}}$, which equals $\operatorname{span}\{z_n\}_{n\in\mathbb{N}}$, which is dense in H. Therefore, $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal sequence, so it is, by definition, an orthonormal basis for H. □

Definition 3.46 (Fourier Transform on $L^1(\mathbb{R})$ [1])**.** The *Fourier transform* of $f \in L^1(\mathbb{R})$ is the function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\,\mathrm{d}x\,, \qquad \text{for } \xi \in \mathbb{R}$$

Theorem 3.37 (Riemann - Lebesgue Lemma[1])**.** *If $f \in L^1(\mathbb{R})$, then $\widehat{f} \in C_0(\mathbb{R})$.*

Definition 3.47 (Inverse Fourier Transform on $L^1(\mathbb{R})$ [1])**.** The inverse Fourier transform of $f \in L^1(\mathbb{R})$ is

$$\check{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{2\pi i\xi x}\,\mathrm{d}x\,, \qquad \text{for } \xi \in \mathbb{R}$$

Theorem 3.38 (Inversion Formula [1])**.** *If $f, \widehat{f} \in L^1(\mathbb{R})$, then both f and $\widehat{f}$ are continuous, and*

$$f(x) = \left(\widehat{f}\right)\check{}(x) = \int_{-\infty}^{\infty} \widehat{f}e^{2\pi i\xi x}\,\mathrm{d}\xi\,, \qquad \textit{for every } x \in \mathbb{R}.$$

Similarly,

$$f(x) = \left(\check{f}\right)\hat{}(x) = \int_{-\infty}^{\infty} \check{f}e^{-2\pi i\xi x}\,\mathrm{d}\xi\,, \qquad \textit{for every } x \in \mathbb{R}.$$

The fact that this works for functions over the real line is very handy, but what if we know more about the functions at hand, specifically, what if the function is periodic or part of the trigonometric system? How does that change the above Fourier Lemmas?

Definition 3.48 (1-periodic functions)**.** Consider the space of functions that are 1-periodic on $\mathbb{R}$ and are square integrable on $[0, 1]$, where 1-*periodic* means that

$$f(x + 1) = f(x) \quad \text{ for } x \in \mathbb{R}.$$

We will denote this space by

$$L^2(\mathbb{T}) = \left\{ f : \mathbb{R} \to \mathbb{C} : f \text{ is 1-periodic and } \int_0^1 |f(x)|^2\,\mathrm{d}x < \infty \right\}.$$

Lemma 3.39 (Riemann - Lebesgue Lemma 2.0[1])**.** *Let $f \in L^1(\mathbb{T})$, then $\widehat{f} \in c_0$, i.e.*

$$\lim_{|n|\to\infty} \widehat{f}(n) = 0.$$

3.4 More Technical Theorems

Theorem 3.40 (Fubini's Theorem for Lebesgue measure [1])**.** *Let E be a measurable subset of $\mathbb{R}^m$ and let F be a measurable subset of $\mathbb{R}^n$. If $f : E \times F \to \bar{\boldsymbol{F}}$ is integrable on $E \times F$ then the following hold.*

1. *$f_x(y) = f(x, y)$ is measurable and integrable on F for a.e. $x \in E$.*
2. *$f^y(x) = f(x, y)$ is measurable and integrable on E for a.e. $y \in F$.*
3. *$g(x) = \int_F f_x(y)\,\mathrm{d}y$ is measurable and integrable on E.*
4. *$h(y) = \int_E f^y(x)\,\mathrm{d}x$ is measurable and integrable on F.*
5. *The following three integrals exist and are finite (i.e., they are real or complex scalars), and they are equal as indicated:*

$$\iint_{E\times F} f(x,y)(\mathrm{d}x\,\mathrm{d}y) = \int_F \left(\int_E f(x,y)\,\mathrm{d}x \right) \mathrm{d}y = \int_E \left(\int_F f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

Theorem 3.41 (Fubini's Theorem for general product measure [2])**.** *Let $(X, \mathscr{I}, \mu)$ amd $(Y, \mathscr{J}, \lambda)$ be a σ-finite measure spaces, and let f be an $(\mathscr{I} \times \mathscr{J})$-measurable function on $X \times Y$.*

1. *If $0 \le f \le \infty$, and if*

$$\phi(x) = \int_Y f_x\,\mathrm{d}\lambda\,, \quad \psi(y) = \int_X f^y\,\mathrm{d}\mu \quad (x \in X, y \in Y),$$

then ϕ is $\mathscr{I}$-measurable, ϕ is $\mathscr{J}$-measurable, and

$$\int_X \phi\,\mathrm{d}\mu = \int_{X\times Y} f\,\mathrm{d}(\mu\times\lambda) = \int_Y \phi\,\mathrm{d}\lambda\,.$$

2. *If f is complex and if*

$$\phi^*(X) = \int_Y |f|_x\,\mathrm{d}\lambda \quad \text{and} \quad \int_X \phi^*\,\mathrm{d}\mu < \infty,$$

then $f \in L^1(\mu \times \lambda)$.

3. *If $f \in L^1(\mu \times \lambda)$, then $f_x \in L^1(\lambda)$ for a.e.$x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$; the functions ϕ and ψ, defined by 1 a.e. are in $L^1(\mu)$ and $L^1(\lambda)$, respectively, and 1 holds.*

Remark 3.10. The first and last integrals in 1 can also be written in the more usual form

$$\int_X \mathrm{d}\mu(x) \int_Y f(x,y)\,\mathrm{d}\lambda(y) = \int_Y \mathrm{d}\lambda(y) \int_X f(x,y)\,\mathrm{d}\mu(x)\,.$$

These are the so-called "iterated integrals" of f. The middle integral in 1 is often referred to as a "double integral".

The combination of this theorem gives the following useful result: If f is $(\mathscr{I} \times \mathscr{J})$-measurable and if

$$\int_X \mathrm{d}\mu(x) \int_Y |f(x,y)| \,\mathrm{d}\lambda(y) < \infty,$$

then the two iterated integrals are finite and equal.

Theorem 3.42 (Lebesgue Differentiation Theorem [1]). *If f is locally integrable on $\mathbb{R}^d$, then for a.e. $x \in \mathbb{R}^d$, we have*

$$\lim_{h\to 0} \frac{1}{|B_h(x)|} \int_{B_h(x)} |f(x) - f(t)| \,\mathrm{d}t = 0$$

and

$$\lim_{h\to 0} \widetilde{f}_h(x) = \lim_{h\to 0} \frac{1}{|B_h(x)|} \int_{B_h(x)} f(t) \,\mathrm{d}t = f(x)$$

Theorem 3.43 (Fundamental Theorem of Calculus [1]). *If $f : a, b \to \mathbb{C}$, then T.F.A.E:*

1. $f \in AC[a, b]$
2. *There exists a function $g \in L^1[a, b]$ such that*

$$f(x) - f(a) = \int_a^x g(t) \,\mathrm{d}t\,, \quad \forall x \in [a, b]$$

3. *f is differentiable a.e. on $[a, b], f' \in L^1[a, b]$, and*

$$f(x) - f(a) = \int_a^x f'(t) \,\mathrm{d}t\,, \quad \forall x \in [a, b]$$

Theorem 3.44 (Fundamental Theorem of Calculus [2]). *Let $I = [a, b]$, let $f : I \to \mathbb{R}^1$ be continuous and nondecreasing. Each of the following three statements about f implies the other two:*

1. *f is AC on I.*
2. *f maps sets of measure 0 to sets of measure 0*
3. *f is differentiable a.e. on I, $f' \in L^1$, and*

$$f(x) - f(a) = \int_a^x f'(t) \,\mathrm{d}t \qquad (a \le x \le b).$$

Theorem 3.45 (Radon-Nikodym Theorem [2]). *Let μ be a positive σ-finite measure on a σ-algebra $\mathfrak{M}$ in a set X, and let λ be a complex measure on $\mathfrak{M}$.*

1. *There is a unique pair of complex measures λ_a and λ_s on $\mathfrak{M}$ such that*

$$\lambda = \lambda_a + \lambda_s, \qquad \lambda_a \ll \mu, \lambda_s \perp \mu.$$

If λ is positive and finite, then so are λ_a and λ_s.

2. *There is a unique* $h \in L^1(\mu)$ *such that*

$$\lambda_a(E) = \int_E h \, \mathrm{d}\mu$$

for every set $E \in \mathfrak{M}$.

Remark 3.11. The pair (λ_a, λ_s) is called the *Lebesgue decomposition* of λ relative to μ. It is a unique decomposition. The existence is the more important part of the first part. The uniqueness of h is immediate. If $h \in L^1(\mu)$, the integral in the theorem defines a measure on $\mathfrak{M}$, which is absolutely continuous with respect to μ. The point is the converse that $\lambda \ll \mu$ (in which case $\lambda_a = \lambda$) is obtained in this way. The function h is called the *Radon-Nikodym derivative* of λ_a with respect to μ. We may also write in the form $\mathrm{d}\lambda_a = h \, \mathrm{d}\mu$, or even in the form $h = \frac{\mathrm{d}\lambda_a}{\mathrm{d}\mu}$.

3.5 Techniques

References

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