MATH 190: MATHEMATICS IN SUSTAINABILITY
Emory University
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Prof. Mitchell Scott

Homework # 2: IESDS, Nash, and Subgame Perfect Equilibria

Solutions

Do 4 out of the first 5 questions. You MUST do question 6 and 7.

1. **Iterated Elimination** In the following normal-form game, which strategy profiles survives each round of iterated elimination of strictly dominated strategies.

		Player 2		
		L	C	R
Player 1	U	6, 8	2, 6	8, 2
	M	8, 2	4, 4	9, 5
	D	8, 10	4, 6	6,7

   Solution. First, we see that Player 1 would never play U as it is strictly dominated by M, since $8 > 6, 4 > 2, 9 > 8$, for Player2 playing L, C, R, respectively. Because Player 1 will never play U, this means that Player 2 will never play C as R strictly dominates it ($5 > 4$ and $7 > 6$ for Player 1 playing M and D respectively).

   Since we now, have a 2x2 game, we see that for Player 1, it is always as good, if not better to choose M over D, since $8 \geqslant 8$, and $9 \geqslant 6$. This means Player 2 will play R as R strictly dominates L ($5 > 2$).

   The IESDS equilibrium is then (M, R) with a payoff of $(9, 5)$.

2. **Popsicle Stands:** Five lifeguard towers are lined up along a beach; the leftmost tower is number 1 and the rightmost tower is number 5. Two vendors, players 1 and 2, each have a popsicle stand that can be located next to one of five towers. There are 25 people located next to each tower, and each person will purchase a popsicle from the stand that is closest to him or her. That is, if player 1 locates his stand at tower 2 and player 2 at tower 3, then 50 people (at towers 1 and 2) will purchase from player 1, while 75 (from towers 3, 4, and 5) will purchase from vendor 2. Each purchase yields a profit of $1.

(a) Specify the strategy set for each player.

Solution. Both players have the same strategy set: They can place their popsicle stand at Towers 1, 2, 3, 4, and 5.

(b) Are there any strictly dominated strategies?

Solution. First we construct the full payoff matrix Since, this game

Player 1 \ Player 2	1	2	3	4	5
1	62.5, 62.5	25, 100	37.5, 82.5	50, 75	62.5, 62.5
2	100, 25	62.5, 62.5	50, 75	62.5,62.5	75,50
3	87.5,37.5	75, 50	62.5, 62.5	75,50	87.5, 37.5
4	75, 50	62.5, 62.5	50,75	62.5,62.5	100, 25
5	62.5, 62.5	50, 75	37.5, 82.5	25, 100	62.5, 62.5

is symmetric, it suffices to see what Player 1 will do. Choosing stand 1 is strictly dominated by choosing stand 2. Similarly, choosing stand 5 is strictly dominated by choosing stand 4. With these eliminated, we see that both choosing stand 2 or 4 is strictly dominated by choosing stand 3. This means the IESDS solution is $(3, 3)$.

(c) Find the set of strategies that survive rationalizability.

Solution. Because choosing stand 1 or 5 are strictly dominated, there are never a best response, i.e. there is always a better strategy to pick for any beliefs one has about their opponent. Similarly, after 1 and 5 are eliminated, we see that choosing either 2 or 4 are strictly dominated by choosing 3, which means the only strategy that survives rationalizability is $\{3\}$.

3. **Public Good Contribution:** Three players live in a town, and each can choose to contribute to fund a streetlamp. The value of having the streetlamp is 3 for each player, and the value of not having it is 0. The mayor asks each player to contribute either 1 or nothing. If at least two players contribute then the lamp will be erected. If one player or no players contribute then the lamp will not be erected, in which case any person who contributed will not get his money back.

   Using the payoff matrix from HW # 1, what outcomes can be supported as pure-strategy Nash equilibria?

   Solution. Recall that the payoff matrices from HW#1 for this problem are: P 3 does NOT contribute (N):

		Player 2 N	Player 2 C
Player 1	N	$\underline{0, 0, 0}$	0, -1, 0
	C	-1, 0, 0	$\underline{2, 2, 3}$

P 3 DOES contribute (C):

		Player 2 N	Player 2 C
Player 1	N	0, 0, -1	$\underline{3, 2, 2}$
	C	$\underline{2, 3, 2}$	2, 2, 2

As we see from underlining the best responses for each player, there are four Nash Equilibria: $(N, N, N), (C, C, N), (C, N, C)$ and (N, C, C). If you aren't convinced because this is a 3 person game, we can also, consider Player 3's best responses between the two matrices.

4. **Discrete All-Pay Auction:** In Section 6.1.4 we introduced a version of an all- pay auction that worked as follows: Each bidder submits a bid. The highest bidder gets the good, but all bidders pay their bids. Consider an auction in which player 1 values the item at 3 while player 2 values the item at 5. Each player can bid either 0, 1, or 2. If player i bids more than player j then i wins the good and both pay. If both players bid the same amount then a coin is tossed to determine who gets the good, but again both pay

   (a) Write down the game in matrix form.

   Solution. First, we have to construct the payoff matrix. We see that if there is a tie, there is a 50% chance of winning and a 50% chance of getting nothing, but still having to pay.

Player 1 \ Player 2	0	1	2
0	1.5,2.5	0,4	0,3
1	2,0	0.5, 1.5	-1,3
2	1,0	1,-1	-0.5, 0.5

   (b) Which strategies survive IESDS?

   Solution. For player 2, we see that bidding 2 dominates bidding 0. Now we see that since 0 is elminated, player 1 would never bid 1 because it is strictly dominated by 2. This leads to the reduced game: We see that we cannot eliminate anymore, so all strategies

Player 1 \ Player 2	1	2
0	0,4	0,3
2	1,-1	-0.5, 0.5

   survive IESDS.

(c) Find the Nash equilibria for this game

Solution. We start from the reduced game (because strictly dominated strategies are never a best response): Putting Player 1's

Player 1 \ Player 2	1	2
0	$\overline{0, 4}$	$\underline{0, 3}$
2	$\underline{1, -1}$	$\overline{-0.5, 0.5}$

best responses in blue underline, and Player 2's best responses in red overline, we see that there is not an instance where both players are playing a best response against each other. This means there is no pure strategy nash equilibrium.

We now compute the mixed-strategy nash equilibrium. Let $\sigma_1 = (p, 1-p), \sigma_2 = (q, 1-q)$ denote the strategy sets of both players. This means for player 2, their payoffs are $4p-(1-p)$ for choosing 1, and $3p+0.5(1-p)$ for choosing 2. For player 2 to be indifferent between these two options, choosing 1 and choosing 2 have to equal each other.

$$\begin{aligned} 4p - 1 + p &= 3p + 0.5 - 0.5p \\ 2.5p &= 1.5 \\ p &= \frac{3}{5} \end{aligned}$$

This means the best response profile for player 2 is

$$\mathrm{BR}_2(\sigma_1) = \begin{cases} 1 & p > \frac{3}{5} \\ 2 & p < \frac{3}{5} \\ 0 \leqslant q \leqslant 1 & p = \frac{3}{5} \end{cases}$$

Now we see that the expected payoff for player 1 picking 0 is 0 and for picking 2 is $q - 0.5(1-q)$. This means player 1 is ambivalent when $0 = -0.5 + 1.5q \implies q = \frac{1}{3}$.

This means the best response profile for player 1 is

$$\mathrm{BR}_1(\sigma_2) = \begin{cases} 0 & q < \frac{1}{3} \\ 2 & q > \frac{1}{3} \\ 0 \leqslant p \leqslant 1 & q = \frac{1}{3}. \end{cases}$$

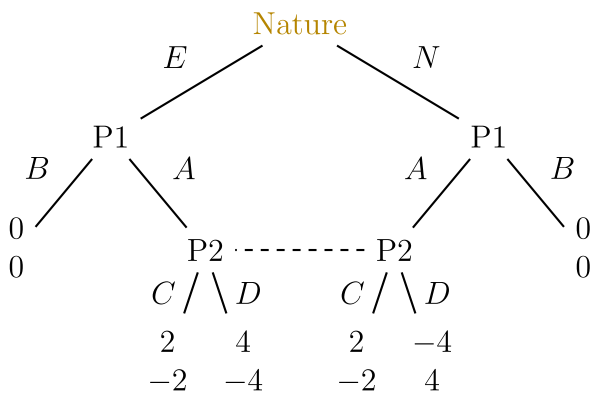
This means that the only time these two players are playing their best response against each other is when $\sigma_1 = \left(\frac{3}{5}, \frac{2}{5}\right)$ and $\sigma_2 =$

$\left(\frac{1}{3}, \frac{2}{3}\right)$ for this reduced game. To get the mixed strategy nash equilibrium for the full game we see:

$$\sigma_1 = \left(\frac{3}{5}, 0, \frac{2}{5}\right) \quad \sigma_2 = \left(0, \frac{1}{3}, \frac{2}{3}\right)$$

5. **The Dean's Dilemma** A student stole the DVD player from the student lounge. The dean of students (player 1) suspects the student (player 2) and begins collecting evidence. However, evidence collection is a random process, and concrete evidence will be available to the dean only with probability $\frac{1}{2}$. The student knows that the evidence-gathering process is under way but does not know whether the dean has collected evidence or not. The game proceeds as follows: The dean realizes whether or not he has evidence and can then choose his action, whether to accuse the student (A) or bounce the case (B) and forget it. Once accused, the student has two options: he can either confess (C) or deny (D). Payoffs are realized as follows: If the dean bounces the case then both players get 0. If the dean accuses the student and the student confesses, the dean gains 2 and the student loses 2. If the dean accuses the student and the student denies, then payoffs depend on the evidence: If the dean has no evidence then he loses face, which gives him a loss of 4, while the student gains glory, which gives him a payoff of 4. If, however, the dean has evidence then he is triumphant and gains 4, while the student is put on probation and loses 4.

   (a) Draw the game tree that represents the extensive form of this game.



   Solution.

   (b) Write down the matrix that represents the normal form of the extensive form you drew in (a).

Solution. First, one has to realize that the Dean knows whether the evidence has been collected or not, so the Dean has two options (accuse A and bounce B) for each information set (with evidence$E$ or without evidence N), this means the Dean has four pure strategies: AA, AB, BA, and BB, where the first is if the evidence is there.

On the other hand, the student doesn't know whether the evidence hunt was successful or not, so they only have one information set with two choices confess C or deny D.

		Student (P2) C	D
Dean (P1)	AA	2, -2	0, 0
	AB	1, -1	2, -2
	BA	1, -1	-2, 2
	BB	0, 0	0, 0

(c) Solve for the Nash equilibria of the game.

Solution. We see that BB and BA are both strictly dominated by AB. This means that we have a reduced game of Since now there

		Student (P2) C	D
Dean (P1)	AA	2, -2	0, 0
	AB	1, -1	2, -2

is no pure strategy Nash Equilbrium, we seek a mixed strategy Nash equilbrium. Let p be the probability the Dean (Player 1) picks AA. Similarly, let q be the probability the student (Player 2) chooses to confess.

For player 2 to be indifferent to confessing, $(-2)p+(-1)(1-p) = 0p+(-2)(1-p)$ must be satisfied, so $-p-1 = 2p-2 \implies p = \frac{1}{3}$, which means that

$$\mathrm{BR}_2\left((p, 1-p)\right) = \begin{cases} C & p < \frac{1}{3} \\ D & p > \frac{1}{3} \\ 0 \leqslant q \leqslant 1 & p = \frac{1}{3}. \end{cases}$$

Similarly, for player 1 to be indifferent to their choice, $2q + 0(1-q) = 1q + 2(1-q) \implies q = \frac{2}{3}$, which means

$$\mathrm{BR}_1\left((q, 1-q)\right) = \begin{cases} AA & q > \frac{2}{3} \\ AB & q < \frac{2}{3} \\ 0 \leqslant p \leqslant 1 & q = \frac{2}{3}. \end{cases}$$

This means the only time these two players are playing a best response against each other is when $p = \frac{1}{3}, q = \frac{2}{3}$.

6. **A Grad Student's Dilemma:** A graduate student is teaching a First Year Seminar class. They have been given $250 to support community within their class. Provide some possible actions in their action set. Describe how these actions might affect the students' payoffs.

   Solution. Answers may vary. Whatever you would be interested in doing to build cameraderie.

7. **Reflection** How long did this assignment take you (be honest)? Do you feel that this is too short or long? How is the material going? How can I help you?

   Solution. Answers will vary.