MATH 190: MATHEMATICS IN SUSTAINABILITY
Emory University
Spring 2026
Prof. Mitchell Scott

Homework # 1: Decision Problems and Static Games

Solutions

1. **Fruit Trees** You have room for up to two fruit-bearing trees in your garden. The fruit trees that can grow in your garden are either apple, orange, or pear. The cost of maintenance is \$100 for an apple tree, \$70 for an orange tree, and \$120 for a pear tree. Your food bill will be reduced by \$130 for each apple tree you plant, by \$145 for each pear tree you plant, and by \$90 for each orange tree you plant. You care only about your total expenditure in making any planting decisions.

   (a) What is the set of possible actions and related outcomes?

   Solution. You have two plots of land. In each you can choose to plant an apple, pear, or orange tree, or leave it barren. Since we don't distinguish between if you plant an apple tree (left) and pear (right) or pear (left) and apple (right) – we consider them both apple and pear. This means you have 10 options [1]. Therefore the actions are to plant:

   i. no trees,
   ii. one apple tree,
   iii. one pear tree,
   iv. one orange tree,
   v. two apple trees,
   vi. two pear trees,
   vii. two orange trees,
   viii. one apple and one pear,
   ix. one apple and one orange,

[1] if you are familiar with combinations, this is the same as choosing 2 out of 4 options with replacements so $\binom{4+2-1}{2} = \frac{(4+2-1)!}{2!(4-2)!} = \frac{5\times 4}{2} = 10$

x. one pear and one orange

The outcomes are having whatever you planted.

(b) What is the payoff of each action/outcome?

Solution. We first notice that our net payoffs are the food savings - the cost of maintenence. As we see in Table **??**, the net payoffs

Choice	**Food Savings**	**Cost**	**Net Payoff**
no trees	0	0	\$0
one apple tree	130	100	\$30
one pear tree	145	120	\$25
one orange tree	90	70	\$20
two apple trees	260	200	\$60
two pear trees	290	240	\$50
two orange trees	180	140	\$40
one apple and one pear	275	220	\$55
one apple and one orange	220	170	\$50
one pear and one orange	235	190	\$45

are computed in the right most column.

(c) Which actions are dominated?

Solution. The two apple trees have a payoff of \$60. This means that all payoffs less than this are dominated.

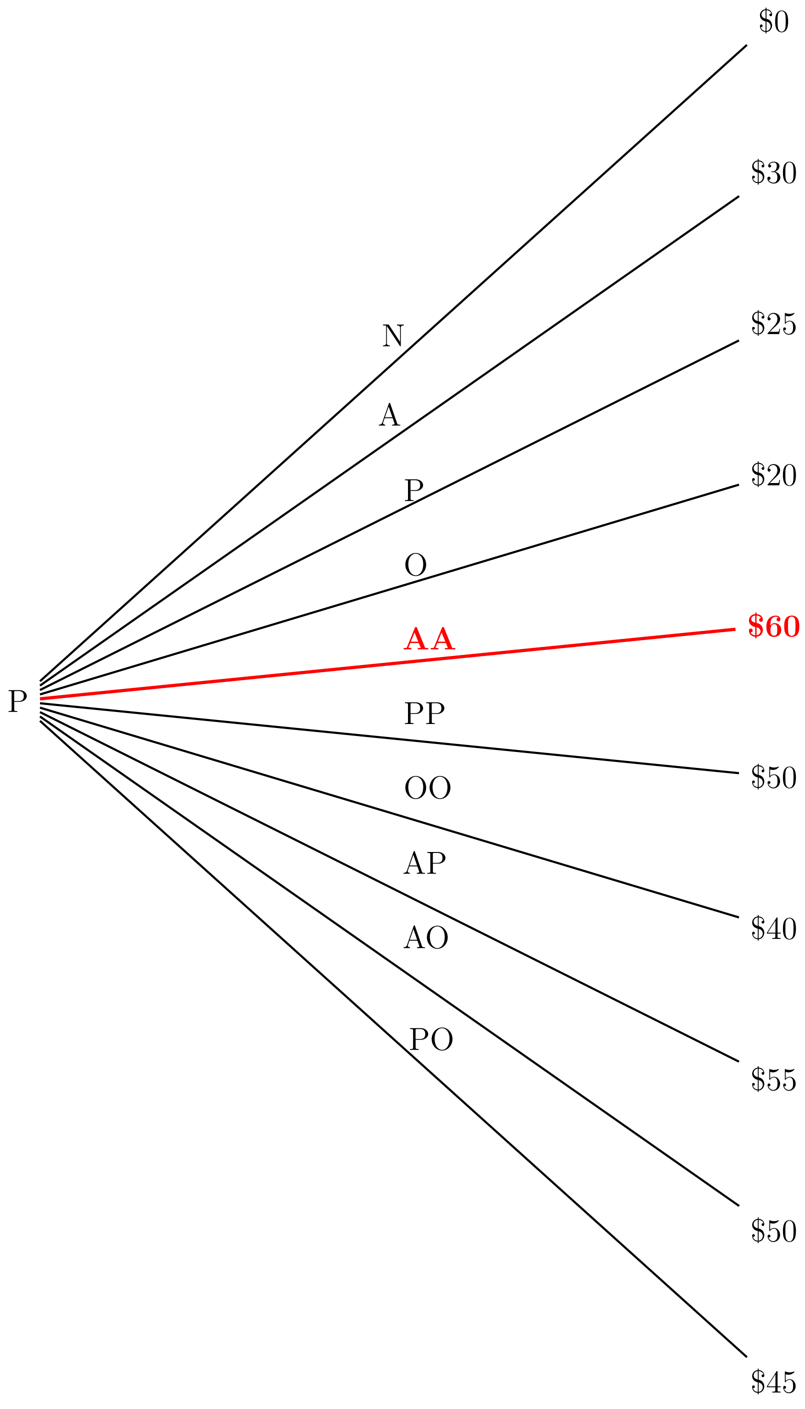
(d) Draw the associated decision tree. What will a rational player choose?

Solution. From the decision tree (Figure **??**), we see that the player should plant two apple trees to get a maximum payoff of \$60.

(e) Now imagine that the reduction in your food bill is half for the second tree of the same kind. (You like variety.) That is, the first apple tree still reduces your food bill by \$130, but if you plant two apple trees your food bill will be reduced by \$130 + \$65 = \$195, and similarly for pear and orange trees. What will a rational player choose now?

Solution. Given the updated table (Tab. **??**), we see that one apple and one pear now have the highest payoff of \$55.

2. **Discount Prices** A local department store sells products at a given initial price, and every week a product goes unsold, its price is discounted by 25% of the original price. If it is not sold after four weeks,


$0
$30
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Choice	Food Savings	Cost	Net Payoff
no trees	0	0	\$0
one apple tree	130	100	\$30
one pear tree	145	120	\$25
one orange tree	90	70	\$20
two apple trees	130 + 65	200	\$-5
two pear trees	145 + 72.5	240	\$-22.5
two orange trees	90 + 45	140	\$-5
one apple and one pear	275	220	\$55
one apple and one orange	220	170	\$50
one pear and one orange	235	190	\$45

Week 1	0.2
Week 2	0.4
Week 3	0.6
Week 4	0.8

it is sent back to the regional warehouse. A set of kitchen knives was just put out for \$200. Your willingness to pay for the knives (your dollar value) is \$180, so if you buy them at a price P, your payoff is $u = 180 - P$. If you don't buy the knives, the chances that they will be sold to someone else, conditional on not having been sold the week before, are as follows

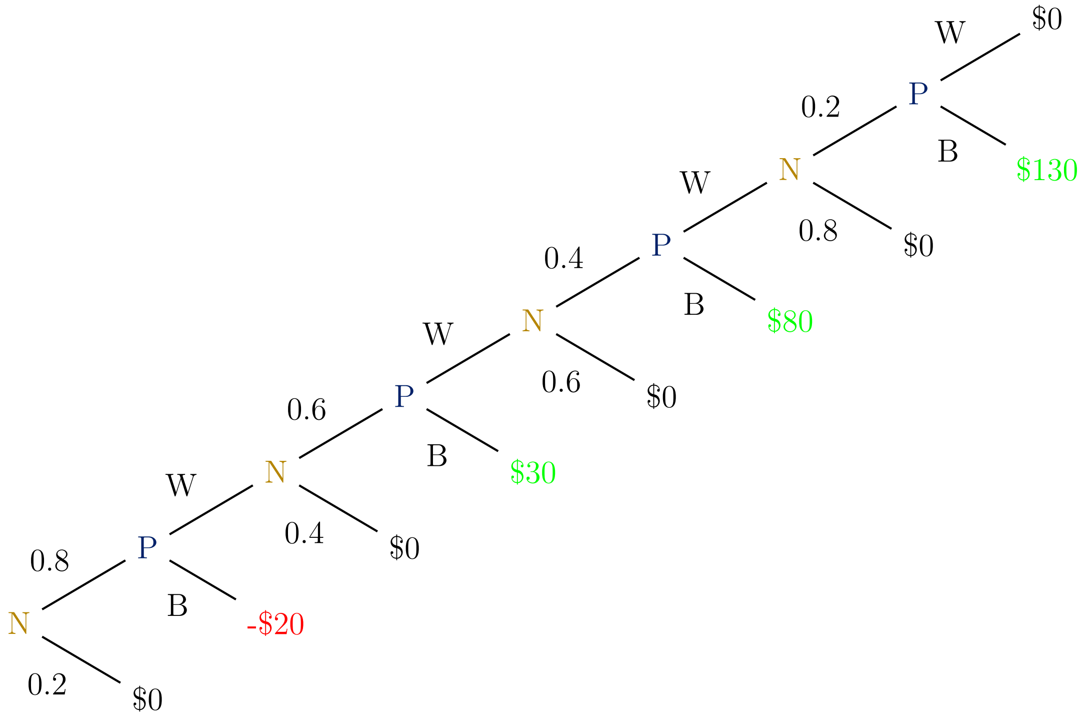
For example, if you do not buy the knives during the first two weeks, the likelihood that they will be available at the beginning of the third week is the likelihood that they do not sell in either week 1 or week 2, which is $0.8 \times 0.6 = 0.48$.

(a) Draw your decision tree for the four weeks after the knives are put out for sale.

Solution. There is a lot to unpack here! First we see that the first week, the knives are set out for \$200. The second week, the knives are discounted 25% of the original price (discounted $\$200 \times 0.25 = \$50$), so they are \$150 at the beginning of week 2, \$100 at the beginning of week 3, and \$50 at the beginning of week 4. Since we value the knives at \$180, our payoff at the beginning of weeks 1,2,3, and 4, are -\$20, \$30, \$80, and \$130, respectively.

We additionally consider Nature (N) as randomizing over the different probabilities. Your decision is to wait (W) of buy (B).

(b) At the beginning of which week, if any, should you run to buy the



knives?

Solution. Just like the problems we do in class, we need to solve by backward induction. If we are the player P, then in week 4, we have the choice to wait with 0 payoff (they will be sent back to the manufacturer) or buy with payoff \$130. As a rational person, you would buy the knives at week 4.

If we wait in week 3, our expected payoff is $0.20 \times (130) = 26$. If we buy in week 3, then our payoff is \$80. Since we are rational, and $80 > 26$, we would buy in week 3.

If we wait in week 2, our expected payoff is $0.4 \times 80 = 32$, while we could buy and get a payoff of 30. Since $32 > 30$, we will wait in week 2, and hope.

Lastly, if we wait in week 1, our expected payoff is $0.6 \times 32 = 19.2$, while we could buy and get a payoff of -20. SInce $19.2 > -20$, we would wait in week 1.

(c) Find a willingness to pay for the knives that would make it optimal to buy at the beginning of the first week.

Solution. If we really value the knives such that at the first week, we have a positive payoff, then that positive payoff is higher than risking it to get more payoff. For example, if we value the knives at \$1,000, then obviously we will buy them at week 4 if they are still around.

Waiting in week 3, we have an expected payoff of $0.2 \times (1000 - 50) = 190$, while buying in week 3 we have a payoff of $1000-100 = 900$. Since $900 > 190$, we will buy in week 3.

Waiting in week 2, we have an expected payoff of $0.4 \times 900 = 360$, while buying in week 2, we have a payoff of $1000 - 150 = 850$. Since $850 > 360$, we would buy in week 2.

Lastly, waiting in week 1, we have an expected payoff of $0.6 \times 850 = 510$, while buying in week 3, we have a payoff of $1000 - 200 = 800$. Since $800 > 510$, we would buy in week 1, which was the objective of the problem.

(d) Find a willingness to pay that would make it optimal to buy at the beginning of the fourth week.

Solution. Similarly to (c) above, to make a late purchase valuable, the willingness to pay must be quite low. Set the willingness to pay at 100. In any week but week 4 the price is above the willingness to pay, so the optimal decision is to wait for week 4 and then buy the knives if they are available.

3. **Matching Pennies** Players 1 and 2 each put a penny on a table simultaneously. If the two pennies come up the same side (heads or tails) then player 1 gets both pennies; otherwise player 2 gets both pennies. Represent this game as a matrix.

   Solution. Let H be a head and T be a tail. Since the winner gets two pennies, but had to risk one, we say that is a payoff of 1. Conversely, the loser, losses one penny with no reward, so a payoff of -1.

		Player 2	
		H	T
Player 1	H	1, -1	-1, 1
	T	-1, 1	1, -1

4. **Public Good Contribution** Three players live in a town, and each can choose to contribute to fund a streetlamp. The value of having the streetlamp is 3 for each player, and the value of not having it is 0. The mayor asks each player to contribute either 1 or nothing. If at least

two players contribute then the lamp will be erected. If one player or no players contribute then the lamp will not be erected, in which case any person who contributed will not get his money back. Write down the normal form of this game.

Solution. The three things we need to list for a Normal form game are the set of players, the strategies, and the payoffs.

(a) The set of players $N = \{1, 2, 3\}$,

(b) The strategy set for each player is $S_1 = S_2 = S_3 = \{N, C\}$, where N means not contributing and C is contributing.

(c) The payoff of different players would be a $2 \times 2 \times 2$ grid of numbers indicating the payoff of player 1, player 2, and player 3.

(d) Payoffs are seen below.

P 3 does NOT contribute (N):

		Player 2	
		N	C
Player 1	N	0, 0, 0	0, -1, 0
	C	-1, 0, 0	2, 2, 3

P 3 DOES contribute (C):

		Player 2	
		N	C
Player 1	N	0, 0, -1	3, 2, 2
	C	2, 3, 2	2, 2, 2

5. **Reflection** How long did this assignment take you (be honest)? Do you feel that this is too short or long? How is the material going? How can I help you?