Homework # 6

Markov Chains: Long Run Behavior and Exits

SP §1.4

1. **Find the stationary distributions for the Markov chains with transition matrices:**

(a)	**1**	**2**	**3**
1	0.5	0.4	0.1
2	0.2	0.5	0.3
3	0.1	0.3	0.6

(b)	**1**	**2**	**3**
1	0.5	0.4	0.1
2	0.3	0.4	0.3
3	0.2	0.2	0.6

(c)	**1**	**2**	**3**
1	0.6	0.4	0.0
2	0.2	0.4	0.2
3	0.0	0.2	0.8

Solution. (a) By subtracting 1 from the diagonal entries in the matrix and letting the last column of the matrix equal 1, the probability transition matrix becomes A, where the last row of the A^{-1} is the stationary distribution, π. The matrix, M, becomes

$$A = \begin{bmatrix} -0.5 & 0.4 & 1 \\ 0.2 & -0.5 & 1 \\ 0.1 & 0.3 & 1 \end{bmatrix}$$

And the last row of the $A^{-1} = \pi$, so

$$\pi = \begin{bmatrix} 0.23404 & 0.40426 & 0.36170 \end{bmatrix} \tag{1}$$

(b) Because the matrix is doubly stochastic, that means that through Theorem 1.14, the stationary distribution, π has a uniform distribution. Since π has to sum to 1 and there are three columns in matrix, M,

$$\pi = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \tag{2}$$

(c) By subtracting 1 from the diagonal entries in the matrix and letting the last column of the matrix equal 1, the probability transition matrix becomes A, where the last row of the A^{-1} is the stationary distribution, π. The matrix, M, becomes

$$A = \begin{bmatrix} -0.4 & 0.4 & 1 \\ 0.2 & -0.6 & 1 \\ 0. & 0.2 & 1 \end{bmatrix}$$

And the last row of the $A^{-1} = \pi$, so

$$\pi = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \end{bmatrix} \tag{3}$$

(a) $\begin{pmatrix} 0.7 & 0.0 & 0.3 & 0.0 \\ 0.6 & 0.0 & 0.4 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.5 \\ 0.0 & 0.4 & 0.0 & 0.6 \end{pmatrix}$ (b) $\begin{pmatrix} 0.7 & 0.3 & 0.0 & 0.0 \\ 0.2 & 0.5 & 0.3 & 0.0 \\ 0.0 & 0.3 & 0.6 & 0.1 \\ 0.0 & 0.0 & 0.2 & 0.8 \end{pmatrix}$ (c) $\begin{pmatrix} 0.7 & 0.0 & 0.3 & 0.0 \\ 0.2 & 0.5 & 0.3 & 0.0 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0.0 & 0.4 & 0.0 & 0.6 \end{pmatrix}$

2. **Find the stationary distributions for the Markov chains on $\{1, 2, 3, 4\}$ with transition matrices:**

Solution. (a) By subtracting 1 from the diagonal entries in the matrix and letting the last column of the matrix equal 1, the probability transition matrix becomes A, where the last row of the A^{-1} is the stationary distribution, π. The matrix, M, becomes

$$A = \begin{bmatrix} -0.3 & 0 & 0.3 & 1 \\ 0.6 & -1 & 0.4 & 1 \\ 0 & 0.5 & -1 & 1 \\ 0 & 0.4 & 0 & 1 \end{bmatrix}$$

And the last row of the $A^{-1} = \pi$, so

$$\pi = \begin{bmatrix} 0.3810 & 0.1905 & 0.1905 & 0.2381 \end{bmatrix} \tag{4}$$

(b) By subtracting 1 from the diagonal entries in the matrix and letting the last column of the matrix equal 1, the probability transition matrix becomes A, where the last row of the A^{-1} is the stationary distribution, π. The matrix, M, becomes

$$A = \begin{bmatrix} -0.3 & 0.3 & 0 & 1 \\ 0.2 & -0.5 & 0.3 & 1 \\ 0 & 0.3 & -0.4 & 1 \\ 0 & 0 & 0.2 & 1 \end{bmatrix}$$

And the last row of the $A^{-1} = \pi$, so

$$\pi = \begin{bmatrix} 0.2105 & 0.3158 & 0.3158 & 0.1579 \end{bmatrix} \tag{5}$$

(c) Because the matrix is doubly stochastic, that means that through Theorem 1.14, the stationary distribution, π has a uniform distribution. Since π has to sum to 1 and there are four columns in matrix, M,

$$\pi = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \tag{6}$$

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3. **Consider the Markov chain with transition matrix:**

	1	**2**	**3**	**4**
1	0.0	0.0	0.1	0.9
2	0.0	0.0	0.6	0.4
3	0.8	0.2	0.0	0.0
4	0.4	0.6	0.0	0.0

(a) **Compute p^2.**

(b) **Find the stationary distributions of p and all of the stationary distributions of p^2.**

(c) **Find the limit of $p^{2n}(x,x)$ as $n \to \infty$.**

Solution. (a) The matrix, p^2, becomes

$$= \begin{bmatrix} 0.44 & 0.54 & 0 & 0 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}$$

(b) Also by doing the method above for finding A and then inverting it, the stationary distribution of p, $\pi = \begin{bmatrix} \frac{8}{30} & \frac{7}{30} & \frac{5}{30} & \frac{10}{30} \end{bmatrix}$. Since p^2 is invertible, there has to be another way to find it. Let's Assume p^2 is made up of four smaller matrices, where p^2=The matrix, p^2, becomes

$$\begin{bmatrix} M_0 & M_1 \\ M_2 & M_3 \end{bmatrix}$$

where $M_1, M_2 =$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which converge to themselves as n approaches infinity and M_0, M_3 become the top left two by two and the bottom right two by two matrix. This means that we can use the familiar method of manipulating M to get it into A, and then inverting it to find the stationary distribution. This gives us two stationary distributions because p^2 is with period of 2, and cannot converge. Let $s = \begin{bmatrix} \frac{8}{15} & \frac{7}{15} & 0 & 0 \end{bmatrix}$, and $t = \begin{bmatrix} 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. The stationary distribution is a linear combination of s and t

(c) Because p is periodic with period of 2, p^{2n} will converge as n goes to infinity. Let P be p^∞. $P(1,1) = \frac{8}{15}, P(2,2) = \frac{7}{15}, P(3,3) = \frac{1}{3}, P(4,4) = \frac{2}{3}$.

4. **Do the following Markov chains converge to equilbrium?**

(a)	**1**	**2**	**3**	**4**
1	0.0	0.0	1.0	0.0
2	0.0	0.0	0.5	0.5
3	0.3	0.7	0.0	0.0
4	1.0	0.0	0.0	0.0

(b)	**1**	**2**	**3**	**4**
1	0.0	1.0	0.0	0.0
2	0.0	0.0	0.0	1.0
3	1.0	0.0	0.0	0.0
4	1/3	0.0	2/3	0.0

(c)	**1**	**2**	**3**	**4**	**5**	**6**
1	0.0	0.5	0.5	0.0	0.0	0.0
2	0.0	0.0	0.0	1.0	0.0	0.0
3	0.0	0.0	0.0	0.4	0.0	0.6
4	1.0	0.0	0.0	0.0	0.0	0.0
5	0.0	1.0	0.0	0.0	0.0	0.0
6	0.2	0.0	0.0	0.0	0.8	0.0

Solution. (a) This chain is closed, irreducible, but when you take the limit to infinity, the entries keep oscillating from zero to nonzero. This implies there is a periodicity of two in the matrix, so it doesn't converge at infinity because Theorem 1.19 assumes I, A, and S, but the A assumption is wrong. This can be seen by finding p, p^2, p^3, p^4. The alternating behavior based on the power mathematically tells me that it doesn't converge at infinity.

(b) This chain is closed, irreducible, but when you try to find the stationary distribution, it is not invertible. Since Theorem 1.19 assumes I, A, and S, but the S assumption is wrong, then this tells me that it doesn't converge at infinity.

(c) This chain is closed, irreducible, but when you take the limit to infinity, the entries keep oscillating from zero to nonzero. Also, looking at the chain set up, this implies there is a periodicity of three in the matrix, so it doesn't converge at infinity because Theorem 1.19 assumes I, A, and S, but the A assumption is wrong. This can be seen by finding p, p^2, p^3, p^4. The alternating behavior based on the power mathematically tells me that it doesn't converge at infinity.

(d) This chain is closed, irreducible, but when you take the limit to infinity, the entries keep oscillating from zero to nonzero. Also, looking at the chain set up, this implies there is a periodicity of three in the matrix, so it doesn't converge at infinity because Theorem 1.19 assumes I, A, and S, but the A assumption is wrong. This can be seen by finding p, p^2, p^3, p^4. The alternating behavior based on the power mathematically tells me that it doesn't converge at infinity.

SP §1.8

5. **A bank classifies loans as paid in full (F), in good standing (G), in arrears (A), or as a bad debt (B). Loans move between the categories according to the following transition probability:**

	F	**G**	**A**	**B**
F	1.0	0.0	0.0	0.0
G	0.1	0.8	0.1	0.0
A	0.1	0.4	0.4	0.1
B	1.0	0.0	0.0	1.0

What fraction of loans in good standing are eventually paid in full? What is the answer for those in arrears?

Solution. (a) Expected Time The expected time for all of these calculations is the percent chance of that row dotted with the duration of that loan. For example, for the variable loan $< Vloan >= 0.55*V + 0.35*30 + 0.00*15 = 15.5 + 0.55*V$ years. The same process is done for the 30 year loan: $< 30loan >= 0.15*V + 0.54*30 + 0.25*15 = 19.95 + 0.15*V$ years. The same process is done for the 15 year loan: $< 15loan >= 0.20*V + 0.00*30 + 0.75*15 = 11.25 + 0.20*V$ years.

(b) Probability of Payment The probability of payment is just $1 - P(foreclosure)$ So the $P_V(paid) = 1 - 0.05 = 0.95$, $P_{30}(paid) = 1 - 0.01 = 0.99$, $P_{15}(paid) = 1 - 0.01 = 0.99$.

6. **Use the second solution in Example 1.48 to compute the expected waiting times for the patterns HHH, HHT, HTT, and HTH. Which pattern has the longest waiting time? Which ones achieve the minimum value of 8?**

Solution. (a) Since every exchange is performed independently over all other exchanges and depends only on the current distribution of balls, this is a Markov Chain.

(b) To describe the probability transition matrix, the state space, S, needs to be established. Each bucket needs to contain at most $b-m$ black balls and at least b balls. This means $S = \{max\{0, b-m\}, min\{b, m\}\}$ The way the probability distribution matrix is constructed is similar to the black and white ball question from last homework. For each state, find the probability of a desired transition, and multiply it with the probability from the other urn.

(c) We need to show that π satistfies the detailed balance condition. Since the only transition that can occur is with the closest neighbors all we have to check is $\pi(i)P(i, i+1) = \pi(i+1)P(i+1, i)$ for each $i \in S$ such that $i+1 \in S$. By pulling out some common factors and rough calculations, we see that they are equal. Therefor π satisfies the detailed balance condition so it is a stationary distribution.