Emory University
Final Exam Review: L'Hopitals, Linear Approximation, and Curve Sketching
MA-111 Calculus I
Date: December 15, 2025
Instructor: Mitchell Scott

Student ID ____________ Name: __________________________

Please read the following instructions carefully.

- This is a review sheet, so there is no time limit. This question booklet contains 8 questions, 14 pages (including the cover) for the total of 80 points/marks. Check to see if any pages are missing. DO NOT scribble or do rough work or make any stray marks on it. Use separate sheet for rough work.
- All the questions are compulsory and all the notations have their usual meaning. You can use standard results directly by writing its statement appropriately. **Read the instructions for individual questions carefully** before answering the questions.
- These questions were developed by me and not overseen by other instructors or the head coordinator.

Question	Points	Score
1	4	
2	5	
3	7	
4	13	
5	15	
6	5	
7	4	
8	27	
Total:	80	

1. (4 points) Which of the following are "indeterminate forms"? (*Hint:* there are 4.)
   - ☐ $1/0$
   - ☐ $0/\infty$
   - √ $0/0$
   - ☐ $\infty/0$
   - √ ∞/∞
   - √ $0 \cdot \infty$
   - ☐ 0^∞
   - √ $\infty - \infty$

2. (5 points) Compute the following limit:

$$\lim_{x\to\infty} \frac{x^3}{e^{3x}}$$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3}{e^{3x}} &= \frac{\infty}{\infty} \\
&\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{3x^2}{3e^{3x}} = \frac{\infty}{\infty} \\
&\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{6x}{6e^{3x}} = \frac{\infty}{\infty} \\
&\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{6}{12e^{3x}} = \frac{6}{\infty} = 0
\end{aligned}$$

3. Consider the function $r(x) = x \ln\left(\frac{x+67}{x}\right)$

   (a) (2 points) Which of the following is the correct form of the limit $\lim_{x\to\infty} r(x)$?
   A. $\infty \cdot \infty$ B. $\infty \cdot 1$ **C.** $\infty \cdot 0$ D. $\infty \cdot$DNE

   (b) (5 points) Exactly compute $\lim_{x\to\infty} x \ln\left(\frac{x+67}{x}\right)$. Carefully show your work. Anytimes you use L'Hopital's Rule, clearly indicate where you are using it.

Solution: As $x \to \infty$ we see that $\ln\left(1 + \frac{67}{x}\right) \to \ln 1 = 0$. So this is an "$\infty \cdot 0$" limit. To use L'Hopital's we need to rewrite it as either a "$0/0$" or "∞/∞". There is no guidance for which way we should rewrite it; just guess, and if the work doesn't get easier, then try the other way. Let's rewrite it as

$$\lim_{x\to\infty} x \ln\left(\frac{x+67}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x+67}{x}\right)}{\frac{1}{x}} = \frac{0}{0}.$$

Now we can apply L'Hopital's rule so

$$\lim_{x\to\infty} \frac{\ln\left(\frac{x+67}{x}\right)}{\frac{1}{x}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{67}{x}}\left(-67x^{-2}\right)}{-x^{-2}} = \lim_{x\to\infty} \frac{67}{1+\frac{67}{x}} = \frac{67}{1+0} = 67.$$

4. Ash wants to approximate the value of $\sqrt[5]{32.5}$, but they forgot their phone at home, so they cannot ask ChatGPT.

(a) (2 points) What is a function $f(x)$ that will help them to get this approximation?

Solution: There are two possible routes here, either Ash could consider $f(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$ or $g(x) = 32.5^x$. These are the only two options since one assumes the exponent is fixed, so they can move around the base (in this case 32.5) to a nearby number, or the other assumes that the base is fixed and can move around the exponent (in this case $\frac{1}{5}$ ot 0.2).

At this point, we haven't learned how to differentiate exponential functions, and even if we did $g(x) = 32.5^x \implies g'(x) = \ln 32.532.5^x$. Since we want to make this problem simpler, with only fractions, integers, or decimals, ln(32.5) is not an option, so we know that the function must be $f(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$.

(b) (2 points) At which point $(a, f(a))$ should Ash draw their tangent line to do the approximation?

Solution: Since we have already decided that $f(x) = x^{1/5} = \sqrt[5]{x}$, we need to find the a-value that is "nice" to work with such that $f(a) \approx 32.5$. Since you are asked to write this approximation using only integers, decimals, or fractions, it might help to start with seeing what integers to the 5^{th} power are close to 32.5. 1 is not a good option because $1^5 = 1$. However, $2^5 = 32.0$, which is very close to 32.5. In other words $f(32) = \sqrt[5]{32} = 2$, so it seems that the point should be $(32, f(32)) = (32, 2)$.

(c) (4 points) Write the equation of the tangent line to your function in (a) at the point you picked in (b).

Solution: We just need to find the tangent line to get the linear approximation, which will be in the point-slope form.

$$f'(x) = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}}, \quad f(32) = 2 \quad f'(32) = \frac{1}{5(32)^{4/5}} = \frac{1}{5(16)} = \frac{1}{80}.$$

This means that the linear approximation is

$$L(x) = 2 + \frac{1}{80}(x - 32).$$

(d) (2 points) Use your tangent line above to approximate $\sqrt[5]{32.5}$. **Your final answer should be a simple integer, decimal, or fraction.**

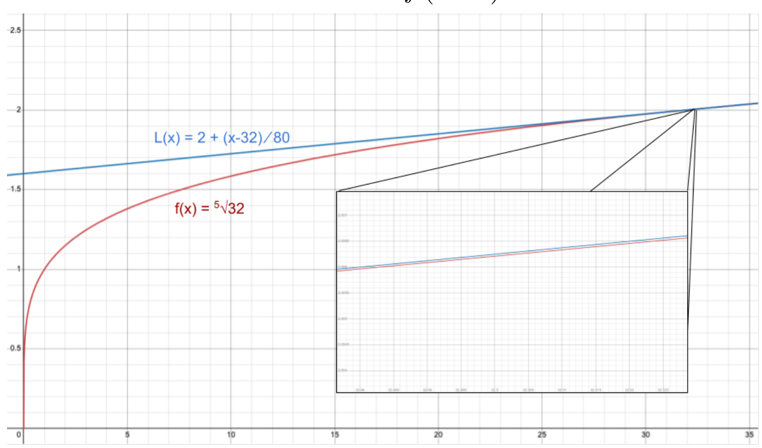
Solution: So when we are plugging in $x = 32.5$, $L(32.5) = 2 + \frac{1}{160}$. One might not be able to know what $\frac{1}{160}$ is however, we do know that $\frac{1}{160} = \frac{1}{16}\frac{1}{10}$. This means once we know what $\frac{1}{16}$ is, we can just move the decimal place over, and we have the answer. Also, we can observe that $\frac{1}{16} = \frac{1}{2^4}$, so it is a half of a half of a half of a half. This means we have $0.5, 0.25, 0.125, 0.0625$, so $\frac{1}{160} = 0.00625$.

(e) (3 points) Ash is a curious cat. Is their estimate an overestimate or an underestimate? Use calculus to justify your reasoning. **A graph is not enough justification here. You need an actual computation.**

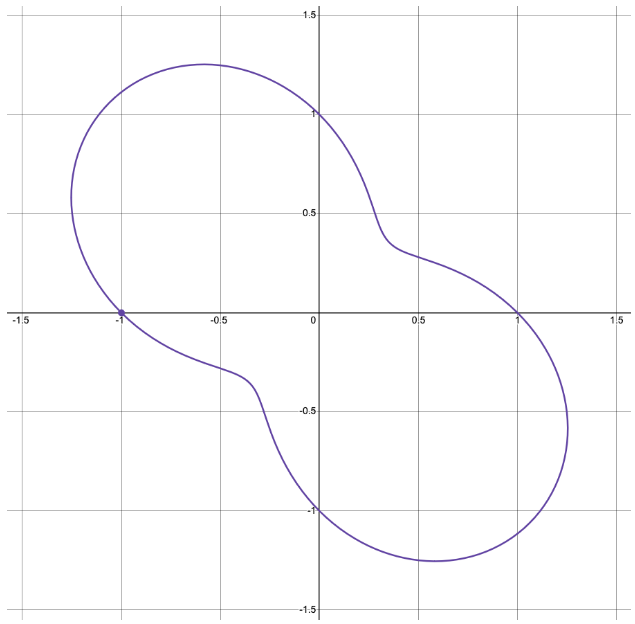
Solution: We lastly want to check if this is an over or under estimation. Doing the calculus calculation of taking the second derivative,

$$f''(x) = \frac{-4}{25x^{9/4}}.$$

Since $x > 0$, we have that this is negative, which means its concave down. In fact, from the graph below, we can see that this linear approximation $L(x) > f(x)$ everywhere except for when $x = 32 \implies L(x) = f(x)$. This means $L(32.5) = 2.00625$ is an over estimate of $f(32.5) = 2.00621129975$.



5. The graph below is given by $x^2 - xy + y^2 = (x^2 + y^2)^{3/2}$ with the point $(-1, 0)$ shown below.



(a) (2 points) What can you say about $\frac{dy}{dx}$ at $(-1, 0)$? **A. It is negative.** B. It is zero. C. It is positive. D. We cannot say anything.

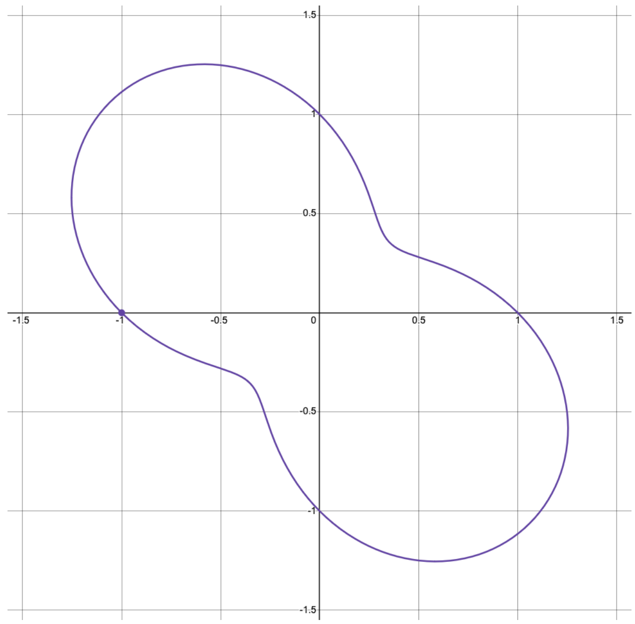
(b) (6 points) Find $\frac{dy}{dx}$ at the point $(-1, 0)$ for the curve $x^2 - xy + y^2 = (x^2 + y^2)^{3/2}$.

Solution:

$$\frac{d}{dx}\left(x^2 - xy + y^2\right) = \frac{d}{dx}\left(\left(x^2 + y^2\right)^{3/2}\right)$$
$$2x - x\frac{dy}{dx} + 2y\frac{dy}{dx} = \frac{3}{2}\left(x^2 + y^2\right)^{1/2} \cdot \left(2x + 2y\frac{dy}{dx}\right)$$
$$(-1)2 - (-1)\frac{dy}{dx} + 0 = \frac{3}{2}\left((-1)^2 + (0)^2\right)^{1/2} \cdot ((-1)2 + 0)$$
$$\frac{dy}{dx} = -3 + 2 = -1$$

(c) (3 points) Which of the following is the equation of the tangent line to this curve at $(-1, 0)$.

A. $L(x) = -\frac{1}{3}(x + 1)$
B. $L(x) = -\frac{1}{2}(x + 1)$
C. $L(x) = -\frac{4}{5}(x + 1)$
D. $L(x) = -(x + 1)$



(d) (2 points) On the curve at input $x = -1.1$, there are two y values: a y value slightly greater than 0, and a y value slightly greater than 1. Use your answer to part (c) to approximate the y-value closer to 0 when $x = -1.1$. **Your final answer should be a simple fration or a simple decimal.**

Solution: The correct answer above had a slope of -1 and went through $(-1, 0)$, so the equation was $L(x) = -x - 1$. This means at $x = -1.1$, we would have $L(-1.1) = -(-1.1) - 1 = 1.1 - 1 = 0.1$.

(e) (2 points) Which of the following is true for your approximation in (d) using linearization?

A. The approximation is an underestimate as the curve is concave up at $(-1, 0)$.

B. The approximation is an underestimate since the linearization is at -1 and $-1 > -1.1$.

C. The approximation is an overestimate as the curve is concave down at $(-1, 0)$.

D. The approximation is an overerestimate since the linearization is at -1 and $-1.1 < -1$.

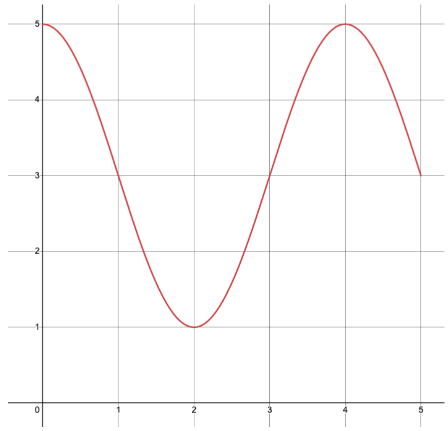
6. (5 points) Evaluate the following limit:

$$\lim_{t\to 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3}$$

Solution: Notice how if $t = 1$, we have $\frac{5-4-1}{10-1-9} = \frac{0}{0}$, so we can do L'Hopital's. We could also try to factor it and cancel, but that looks like an absolute nightmare.

$$\lim_{t\to 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3} \stackrel{\text{L'H}}{=} \lim_{t\to 1} \frac{20t^3 - 8t}{-1 - 27t^2} = \frac{20-8}{-1-27} = \frac{-3}{7}$$

7. Consider $f(x)$ whose graph is given below and let ${}_1A_f(x) = \int_1^x f(t)\,\mathrm{d}t$.



(a) (1 point) What can you say about ${}_1A_f(1) = \int_1^1 f(t)\,\mathrm{d}t$.
   A. ${}_1A_f(1) < 0$
   B. ${}_1A_f(1) = 0$
   C. ${}_1A_f(1) > 0$

(b) (1 point) What can you say about ${}_1A_f(0) = \int_1^0 f(t)\,\mathrm{d}t$.
   A. ${}_1A_f(0) < 0$
   B. ${}_1A_f(0) = 0$
   C. ${}_1A_f(0) > 0$

(c) (1 point) On what interval(s) of x in $(0, 5)$ is ${}_1A_f(x) = \int_1^x f(t)\,\mathrm{d}t$ increasing?
   A. $(0, 2)$ and $(4, 5)$ only
   B. $(2, 4)$ only
   C. $(0, 5)$

(d) (1 point) On what interval(s) of x in $(0, 5)$ is ${}_1A_f(x) = \int_1^x f(t)\,\mathrm{d}t$ concave up?
   A. $(0, 2)$ and $(4, 5)$ only
   B. $(2, 4)$ **only**
   C. $(0, 5)$

(e) Sketch the graph of ${}_1A_f(x)$ on $[0, 5]$ on the axes below, and **label one x-intercept** with exact values.

Solution: First, we know that the x-intercept is $(0, 1)$ since $\int_1^1 f(t)\,\mathrm{d}t = 0$; this is the only x-intercept because the function ${}_1A_f(x)$ is always adding area since $f(t) > 0$.
Since the function $f(t) > 0$, and because of the property ${}_1A_f(0) = \int_1^0 f(t)\,\mathrm{d}t = -\int_0^1 f(t)\,\mathrm{d}t < 0$, then ${}_1A_f(0) < 0$. By inspection, we see that $\int_0^1 f(t)\,\mathrm{d}t > 4$. Additionally, we know that ${}_1A_f(x)$ is always increasing, but only concave up from (2,4).

Putting these all together, we get:

8. This problem will guide you through the steps to sketch the graph of the following function:

$$h(x) = \frac{x^2 - x - 2}{x^2 - x - 6}.$$

(a) (2 points) What is the domain of $h(x)$?

Solution: Since this is a rational function with familiar functions in the numerator and denominator, we know the only possible points, where the function isn't defined, is when the denominator is zero, so for our case, we can factor the bottom into $x^2 - x - 6 = (x-3)(x+2)$, we have when $(x+2) = 0 \implies x = -2$, and when $(x-3) = 0 \implies x = 3$. This means $f(x)$ has a domain of all real numbers except when $x = -2, 3$.

(b) (2 points) Find the y-intercept of $h(x)$

Solution: The y-intercept of a function is the output of the function when $x = 0$. Plugging this in, we have

$$f(0) = \frac{(0)^2 - (0) - 2}{(0)^2 - (0) - 6} = \frac{-2}{-6} = \frac{1}{3}$$

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(c) (2 points) Find the x-intercept of $h(x)$, if there is one.

Solution: The x-intercept of a function is when $h(x) = 0$. Since this is a rational function, this occurs only when $x^2 - x - 2 = (x-2)(x+1) = 0$. This means the x intercepts are $x = -1, 2$

(d) (2 points) Does $h(x)$ have any discontinuities? If so, what kind and where?

Solution: Discontinuities happen when there are issues with the denominator, such as being 0. This means we will have discontinuities when $x = -2, 3$. We will use limits to determine what kind of discontinuities we have.

$$\lim_{x \to -2^-} h(x) = \frac{(-2)^2 - (^-2) - 2}{((-2) - 3)(0^-)} = \infty, \quad \lim_{x \to -2^+} h(x) = \frac{(-2)^2 - (^-2) - 2}{((-2) - 3)(0^+)} = -\infty.$$

By classifying discontinuities, since both limits increase (decrease) without bound, this is a vertical asymptote. Similarly,

$$\lim_{x \to 3^-} h(x) = \frac{(3)^2 - (3) - 2}{((3) + 2)(0^-)} = -\infty, \quad \lim_{x \to 3^+} h(x) = \frac{(3)^2 - (3) - 2}{((3) + 2)(0^+)} = +\infty.$$

By classifying discontinuties, since both limits decrease (inccrease) without bound, this is a vertical asymptote.

(e) (2 points) What are the horizontal asymptotes of $h(x)$?

Solution: Before we are able to figure out the horizontal asymptotes, we must compute the limits to infinity

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2-x-2}{x^2-x-6} &= \frac{\infty}{\infty}\\ &\overset{\text{L'Hopital}}{=} \lim_{x\to\infty}\frac{2x-1}{2x-1} = \frac{\infty}{\infty}\\ &\overset{\text{L'Hopital}}{=} \lim_{x\to\infty}\frac{2}{2} = 1.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\lim_{x\to-\infty}\frac{x^2-x-2}{x^2-x-6} &= \frac{\infty}{\infty}\\ &\overset{\text{L'Hopital}}{=} \lim_{x\to-\infty}\frac{2x-1}{2x-1} = \frac{-\infty}{-\infty}\\ &\overset{\text{L'Hopital}}{=} \lim_{x\to-\infty}\frac{2}{2} = 1.\end{aligned}$$

This means that the two horizontal asymptotes happen to be the same so the horizontal asymptotes are $y = 1$, both going to the left and the right.

- ☐ Only $h(x) = \infty$
- √ **Only** $h(x) = 1$
- ☐ Both $h(x) = 0$ and $h(x) = \infty$
- ☐ This function has no horizontal asymptotes.

(f) (2 points) Make a sign chart of the **first derivative**. What does it tell you about h?

Solution:
We proceed with the quotient rule:

$$h'(x) = \frac{(x^2-x-6)(2x-1)-(x^2-x-2)(2x-1)}{(x^2-x-6)^2} = \frac{-8x+4}{(x-3)^2(x+2)^2}$$

The points where f' DNE or $h'(x) = 0$ are when $x = -2$ or $x = 0$, respectively; This means that the sign chart would look like:

h	$\nearrow$	DNE	$\nearrow$	$\rightarrow$	$\searrow$	DNE	$\searrow$
h'	$+$	DNE	$+$	0	$-$	DNE	$-$
x		-2		$1/2$		3	

(g) (1 point) Find the point $(x, h(x))$ that is a *local minimum.*

Solution: We know there can be no local min as there is a vertical asymptote at $x = -2$ and $x = 3$. Since both $x = -2^+$ and $x = 3^-$ deccrease without bound, there cannot be a local min.

(h) (1 point) Find the point $(x, h(x))$ that is a *local maximum.*

Solution: Looking at the chart above, we see that $h(x)$ has a local maximum when $x = 1/2$. This is since the derivative is increasing before $x = 0$ and decreasing after.

(i) (3 points) Make a sign chart of the **second derivative**. What does it tell you about h? Are there any points of inflection?

Solution:
We proceed with the quotient rule:

$$\begin{aligned} h''(x) &= \frac{(x-3)^2(x+2)^2(-8) - (-8x+4)(2(x^2-x-6)(2x-1))}{(x-3)^4(x+2)^4} \\ &= \frac{-8(x-3)(x+2) + 32x^2 - 16x + 8}{(x-3)^3(x+2)^3} \\ &= \frac{24x^2 - 24x + 56}{(x-3)^3(x+2)^3} \end{aligned}$$

The points where f'' DNE are when the denominator equals 0, which is when $x = -2, 3$. The points where $h''(x) = 0$ is when $3x^2 - 3x + 7 = 0$. However, if we use the quadratic formula, you will find the only roots are imaginary since $\sqrt{b^2 - 4ac} = \sqrt{9 - 4(3)(7)} = \sqrt{-75}$. We put this in the sign chart below.

h	$\cup$	DNE	$\cap$	DNE	$\cup$
h''	$+$	DNE	$-$	DNE	$+$
x		-2		3	

We see that h is concave up when $x < -2$, concave down between $-2 < x < 3$, and concave up again when $x > 3$. Even though concavity changes at $x = -2, 3$, since these points aren't in the domain of h, they are not considered inflection points.

(j) (10 points) Using the information from the previous parts, sketch the graph of the function:

$$h(x) = \frac{x^2 - x - 2}{x^2 - x - 6}$$

Solution: