Emory University
Final Exam Review: Integration
MA-111 Calculus I
Date: December 10, 2025
Instructor: Mitchell Scott

Student ID _____________ Name: ____________________________

Please read the following instructions carefully.

- **This is simply a review session!** This question booklet contains 3 questions, 11 pages (including the cover) for the total of 34 points/marks. Check to see if any pages are missing. DO NOT scribble or do rough work or make any stray marks on it. Use separate sheet for rough work.
- This is meant to represent what an actual exam might look like. **Read the instructions for individual questions carefully** before answering the questions.
- This was not overseen by the coordinator nor other instructors.

Question	Points	Score
1	12	
2	12	
3	10	
Total:	34	

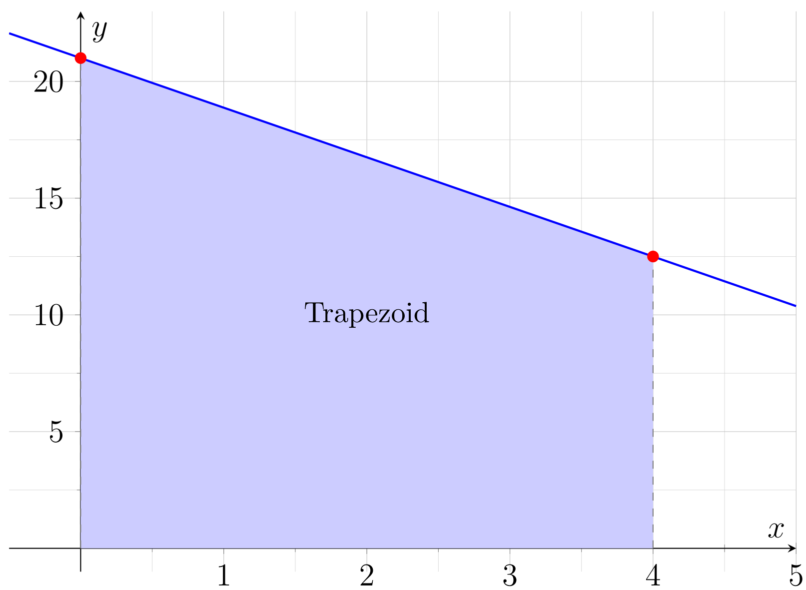
1. Solve the following definite integrals:

   (a) (4 points)

$$\int_0^4 (-2.125x + 21)\,\mathrm{d}x$$

Solution: There are two different lenses that we can use here to compute this definite integral– geometrically and using the fundamental theorem of calculus (FTC).

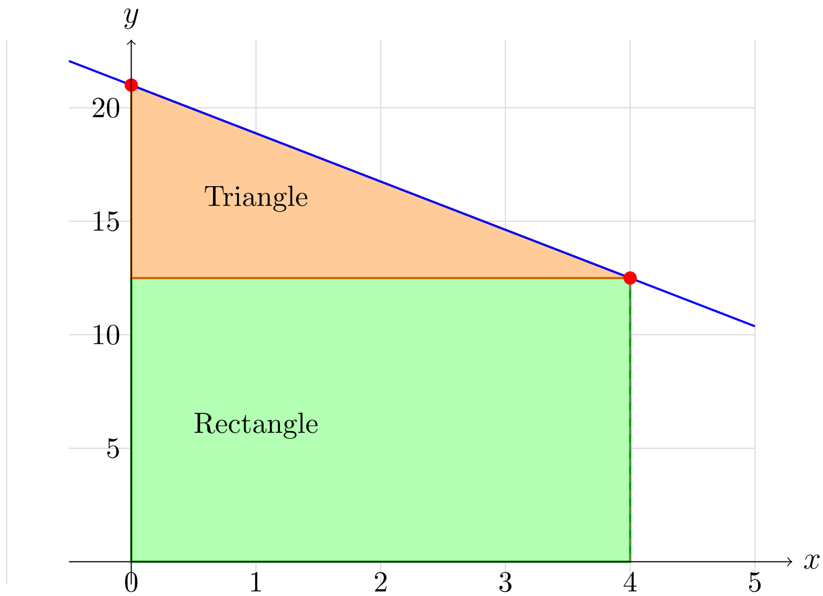
To motivate the geometric way, we must remember that definite integrals are just the area under the curve, and in this case the curve is just the line $-2.125x+21$. We take the area between 0 and 4. This means we have the following shape.



If we recognize this as a trapezoid, then we just have to use the trapezoidal area formula. The height when $x = 0$ is $f(0) = -2.125(0) + 21 = 21$, and the height when $x = 4$ is $f(4) = -2.125(4) + 21 = -8.5 + 21 = 12.5$. This means:

$$A_{\text{trapezoid}} = \left(\frac{21 + 12.5}{2}\right)(4 - 0) = 17.25 * 4 = 67.$$

However, if you recognize this as the sum of a triangle and rectangle, the graph could look like this:



where

$$A_{\text{triangle}} = \frac{1}{2}(4-0)(21-12.5) = 17$$
$$A_{\text{rectangle}} = (4-0)(12.5-0) = 50$$

This means that the total area under the curve is $17 + 50 = 67$, which matches the area found above.

Lastly, we turn our attention to the calculus way of solving this equation. Since we have $-2.125x+21$ that we want to integrate, we can use the constant multiple rule, the integral sum rule, and the inverse power rule to get:

$$\begin{aligned}\int_0^4 -2.125x + 21\,\mathrm{d}x &= -2.125\frac{x^2}{2} + 21x\Big|_{x=0}^{4} \\ &= \left(-2.125\frac{4^2}{2} + 21(4)\right) - \left(-2.125\frac{0^2}{2} + 21(0)\right) \\ &= -2.125(8) + 84 - 0 + 0 = -17 + 84 = 67.\end{aligned}$$

Since we know that this definite integral is 67 from our previous computations, we know that we correctly computed this, but we could also check that the derivative of our antiderivative is the function that we originally wanted.

$$\frac{\mathrm{d}}{\mathrm{d}x} - 2.125\frac{x^2}{2} + 21x = -2.125\frac{2x}{x} + 21.$$

Since we are more comfortable with derivatives, we see that simply applying the power rule, we get our original function back.

(b) (4 points)

$$\int_0^{\ln(1+\pi)} e^x \cos(1-e^x)\,\mathrm{d}x$$

Solution: This doesn't look like any elementary function that we know how to take the antiderivative of, and in fact it looks like the product of two easier functions. Since there is no product or quotient rule for integrals, that leaves u-substitution as the only option. This should make sense as we have a composite function:

$$f(g(x)) = \cos(1-e^x) \implies f(x) = \cos(x), g(x) = 1-e^x.$$

This means let $u = g(x) = 1 - e^x$ which implies $\mathrm{d}u = 0 - e^x\,\mathrm{d}x = -\,\mathrm{d}u = e^x\,\mathrm{d}x$. This means we can rewrite the integral as

$$\int_{x=0}^{x=\ln(1+\pi)} \cos(u)\left(-\,\mathrm{d}u\right).$$

We need to convert the bounds of the integral into the u space. We do that by substituting in the bounds in terms of x into the u. This means

$$u = 1 - e^{\ln(1+\pi)} = 1 - (1+\pi) = -\pi$$
$$u = 1 - e^0 = 1 - 1i = 0$$

Now we get the integral in the final form

$$\int_0^{-\pi} -\cos(u)\mathrm{d}u = -\sin(u)\Big|_0^{-\pi} = -\sin(-\pi) - (-\sin(0)) = -0 + 0 = 0.$$

This was relatively straightforward when you picked the correct u for substitution–the entirety of the sinde o9f the composite function, but what would happen if you didn't? Are you unable to complete the computation? NO! Imagine we have the same integral:

$$\int_0^{\ln(1+\pi)} e^x \cos(1-e^x)\,\mathrm{d}x$$

and we set our $u = e^x$ which would imply that $\mathrm{d}u = e^x\,\mathrm{d}x$. Then if we change the bounds we would have $u\left(\ln(1+\pi)\right) = e^{\ln(1+\pi)} = 1+\pi$ and then $u(0) = e^0 = 1$, which means we can rewrite this as:

$$\int_1^{1+\pi} \cos(1-u)\mathrm{d}u.$$

This results in an easier looking integral, but we are still not able to solve it as it involves another composite function. This means that we can do another round of u-substitution. If we use the inside of the composite function, $w = 1 - u \implies \mathrm{d}w = -\,\mathrm{d}u \implies -\,\mathrm{d}w = \mathrm{d}u$. Then we can rewrite the bounds to be in terms of w, so $w(1+\pi) = 1 - (1+\pi) = -\pi$ and $w(1) = 1 - 1 = 0$. This means we get the following integral:

$$\int_0^{-\pi} \cos(w)\,(-\,\mathrm{d}w) = -\int_0^{-\pi} \cos(w)\,\mathrm{d}w = -\sin(w)\Big|_0^{-\pi} = 0.$$

We see that we end up with the same overall u-substitution $w = 1 - e^x$, but it took two separate rewritings of the integral. This is a good reminder to use the entirety of the inside composite function.

(c) (4 points)

$$\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,\mathrm{d}x$$

Solution: Here we see that u-substitution must be the way forward because we have an composite trigonometric function. Since we see that $\cos(\sin(x))$ is of the form $f(g(x))$, where $f(x) = \cos(x), g(x) = \sin(x)$. Using the advise above, we should make $u = \sin(x)$, which means $\mathrm{d}u = \cos(x)\,\mathrm{d}x$. This works out well for us, because we can then write the integral as

$$\int_{x=-\pi}^{x=\frac{\pi}{2}} \cos(u)\mathrm{d}u.$$

In this answer, we will explore the two different techniques for changing the bounds for u-substitution. First, we will change the bounds of the integral into the u domain, leading to $u(\frac{\pi}{2}) = \sin\big(\frac{\pi}{2} = 1\big)$ and $u(-\pi) = \sin(-\pi) = 0)$, so

$$\int_{x=0}^{x=1} \cos(u)\mathrm{d}u = \sin(u)\Big|_0^1 = \sin(1) - \sin(0) = \sin(1).$$

Alternatively, we could have kept the bound in the x-domain, which means that we need to convert u back to $\sin(x)$ before we substitute the x values. This leads to

$$\begin{aligned}
\int_{x=-\pi}^{x=\frac{\pi}{2}} \cos(u)\mathrm{d}u &= \sin(u)\Big|_{x=-\pi}^{x=\frac{\pi}{2}} \\
&= \sin(\sin(x))\Big|_{-\pi}^{\frac{\pi}{2}} \\
&= \sin\left(\sin\left(\frac{\pi}{2}\right)\right) - \sin(\sin(-\pi)) \\
&= \sin(1) - \sin(0) = \sin(1)
\end{aligned}$$

At the end of the day, we get the same answer, so there is no “right” or “wrong” way to handle it but a matter of personal taste. I like switching everything to the u domain, so that I can just apply the FTCII and not worry about switching back, but whatever you prefer is better for you, you should use :)

2. Solve the following indefinite integrals:

   (a) (4 points)

$$\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,\mathrm{d}x$$

Solution: This is a relatively tricky one. Your first instinct might be to let $u = 1 - x^2$ because you see that as the inside of the composite function. However, this would mean $\mathrm{d}u = -2x\,\mathrm{d}x$, which is a problem because we don't have an x anywhere in the integral. Since we cannot multiply/divide by x, since there is no product or quotient rules for integrals, we have to launch a different approach. Instead we could use our knowledge of inverse trigonometric functions to let $u = \arcsin(x)$ which means that $\mathrm{d}u = \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$. This cleans up the indefinite integral quite nicely, leading to

$$\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \int u^1 \mathrm{d}u = \frac{u^{1+1}}{1+1} + c = \frac{u^2}{2} + c = \frac{\arcsin(x)^2}{2} + c.$$

We can always check our work by taking the derivative of the antiderivative to check our work. Because this was a u substitution problem, the derivative will most likely include the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\arcsin(x)^2}{2} + c &= \frac{2\arcsin(x)}{2}\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x)\\ &= \arcsin(x)\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

(b) (4 points)

$$\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,\mathrm{d}x$$

Solution: This is more straightforward indefinite integral. We see the composite function $\cos(\sqrt{x}) = f(g(x)) \implies f(x) = \cos(x), g(x) = \sqrt{x}$, so we let $u = \sqrt{x} = x^{\frac{1}{2}} \implies \mathrm{d}u = \frac{1}{2}x^{-1/2} = \mathrm{d}x \implies 2\,\mathrm{d}u = \frac{1}{\sqrt{x}}\,\mathrm{d}x$. Now we can rewrite this as

$$\int 2\cos(u)\mathrm{d}u = 2\sin(u) + c = 2\sin\big(\sqrt{x}\big) + c.$$

We now have to check our antiderivative to make sure that we are correct.

$$\frac{\mathrm{d}}{\mathrm{d}x}2\sin\big(\sqrt{x}\big) + c = 2\cos\big(\sqrt{x}\big)\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{2\cos(\sqrt{x})}{2\sqrt{x}} = \frac{\cos(\sqrt{x})}{\sqrt{x}}.$$

(c) (4 points)

$$\int \left(x\cos\left(x^2+1\right) + \frac{x}{x^2+1}\right)\mathrm{d}x$$

Solution: This question is used to test properties of indefinite integrals. The first is that the integral of sums is the sum of integrals. This means we can rewrite this as:

$$\int x\cos\left(x^2+1\right)\mathrm{d}x + \int \frac{x}{x^2+1}\,\mathrm{d}x\,.$$

Now it is easier to see the plan of attack on each integral separately. It looks like each could use u substitution as each function can be viewed as a composite function. (*Hint: how can you write the second integral as a composite function?*) In fact, if we let $u = x^2+1 \implies \mathrm{d}u = 2x\,\mathrm{d}x \implies \frac{\mathrm{d}u}{2} = x\,\mathrm{d}x$, then we get

$$\int \cos(u)\frac{\mathrm{d}u}{2} + \int \frac{1}{u}\frac{\mathrm{d}u}{2}.$$

Now each integral is basic rules with constant multiples.

$$\begin{aligned}\int \cos(u)\frac{\mathrm{d}u}{2} + \int \frac{1}{u}\frac{\mathrm{d}u}{2} &= \frac{1}{2}\sin(u) + \frac{1}{2}\ln|u| + c\\ &= \frac{1}{2}\sin\big(x^2+1\big) + \frac{1}{2}\ln|x^2+1| + c\end{aligned}$$

3. Solve the following integrals:

   (a) (4 points)

$$\int \left(x\cos\left(x^2+1\right) + \frac{1}{x^2+1} \right) \mathrm{d}x$$

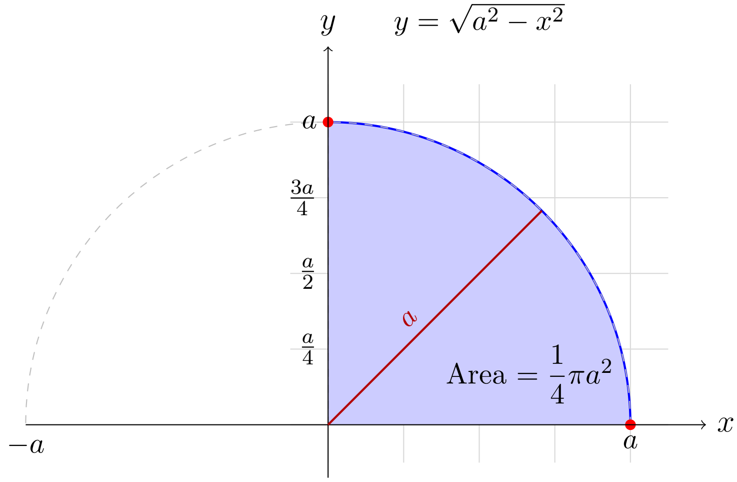
Solution: The last problem we got lucky that both of the integrals involved were u-substitution; however, we can use treat each integral using separate methods. That is what this problem is meant to tackle. For example, it is essentially the exact same integral, but with the exception that we don't have an x in the numerator in the second integral. This means we cannot use the same u since the du term cannot work. However, we should see that the second term is an integral we should know. This means we get:

$$\begin{aligned} \int \cos(u)\frac{\mathrm{d}u}{2} + \int \frac{1}{x^2+1}\,\mathrm{d}x &= \frac{1}{2}\sin(u) + \arctan(x) + c \\ &= \frac{1}{2}\sin\left(x^2+1\right) + \arctan(x) + c \end{aligned}$$

(b) (6 points) Using the trigonometric identity $\cos^2(x) = \frac{1}{2}(\cos(2x)+1)$, solve

$$\int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x$$

Solution: From the integral, we can see that this is the formula for the upper half circle from $x = 0$ being the origin and $x = a$ being the radius. We geometrically know that this is the area of a quarter circle with radiuss a. So overall teh integral should be $\frac{1}{4}a^2$. Let's prove it!



We need to start by factoring out the a^2 from the square root

$$\begin{aligned}\int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x &= \int_0^a \sqrt{a^2\left(1 - \frac{x^2}{a^2}\right)}\,\mathrm{d}x \\ &= \int_0^a a\sqrt{1 - \frac{x^2}{a^2}}\,\mathrm{d}x \\ &= \int_0^a a\sqrt{1 - \left(\frac{x}{a}\right)^2}\,\mathrm{d}x\end{aligned}$$

This next step might be a little confusing at first, but if we let $x = a\sin(t)$, we will be able to use a pythagorean identity to get something that involves $\cos(x)$ which will allow us to use the hint we were given. So since $\frac{x}{a} = \sin(t) \implies \mathrm{d}x = a\cos(t)\,\mathrm{d}t$. This isn't really a u substitution, but rather a different way to compute the derivative, which would have been hard to without trigonometry. Lastly, we need to convert the bounds to get $x = a \implies a = a\sin(t)$, which only happens when $\sin(t) = 1 \implies t = \frac{\pi}{2}$. Also, $x = 0 \implies 0 = a\sin(t) \implies \sin(t) = 0$ since we know $a > 0$ as it is the radius of a circle. This means $t = 0$ so $\sin(t) = 0$. We are also aware that these values are unchanged if we go around another 2π, but since we only care about the angle between $0 \leq t \leq \frac{\pi}{2}$, we are good. Substituting these reparameterizations into the original integral, we get

$$\begin{aligned}\int_0^a a\sqrt{1-\left(\frac{x}{a}\right)^2}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} a^2\sqrt{1-\sin^2(t)}\cos(t)\,\mathrm{d}t\\ &= \int_0^{\frac{\pi}{2}} a^2\cos^2(t)\,\mathrm{d}t\,,\end{aligned}$$

where the last equality comes from the Pythagorean identity

$$1 = \sin^2(t)+\cos^2(t) \implies \sqrt{1-\sin^2(t)} = \sqrt{\cos(t)} = \cos(t),$$

since we already established that t is inbetweenm 0 and $\frac{\pi}{2}$. Now we can use the hint provided to get:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} a^2\cos^2(t)\,\mathrm{d}t &= \int_0^{\frac{\pi}{2}} a^2\frac{1}{2}\left(\cos(2x)+1\right)\mathrm{d}t\\ &= \frac{a^2}{2}\left(\frac{1}{2}\sin(2t)+t\right)\Bigg|_0^{\frac{\pi}{2}}\\ &= \frac{a^2}{2}\left(\frac{1}{2}\sin\left(2\frac{\pi}{2}\right)+\frac{\pi}{2}-\left[\frac{1}{2}\sin(0)+0\right]\right)\\ &= \frac{a^2}{2}\left(\frac{1}{2}\sin\left(2\frac{\pi}{2}\right)+\frac{\pi}{2}+0\right)\\ &= \frac{a^2}{2}\left(0+\frac{\pi}{2}\right) = \frac{a^2\pi}{4},\end{aligned}$$

which is the desired result.