Emory University

Review for Exam # 2

MA-111 Calculus I

Date: October 24, 2025

Instructor: Mitchell Scott

Student ID ____________ Name: ____________________________

Please read the following instructions carefully.

- **This is simply a review session!** This question booklet contains 5 questions, 12 pages (including the cover) for the total of 75 points/marks. Check to see if any pages are missing. DO NOT scribble or do rough work or make any stray marks on it. Use separate sheet for rough work.
- This is meant to represent what an actual exam might look like. **Read the instructions for individual questions carefully** before answering the questions.
- This was not overseen by the coordinator nor other instructors.

Question	Points	Score
1	10	
2	20	
3	15	
4	20	
5	10	
Total:	75	

1. (10 points) **Implicit differentiation.** Lucky the Leprechaun wants to give you a nice token, as he knows some people might be stressed. He concocts the following curve

$$\left(x^2+y^2\right)^3 - 404x^2y^2 - \frac{1}{10000} = 0.$$

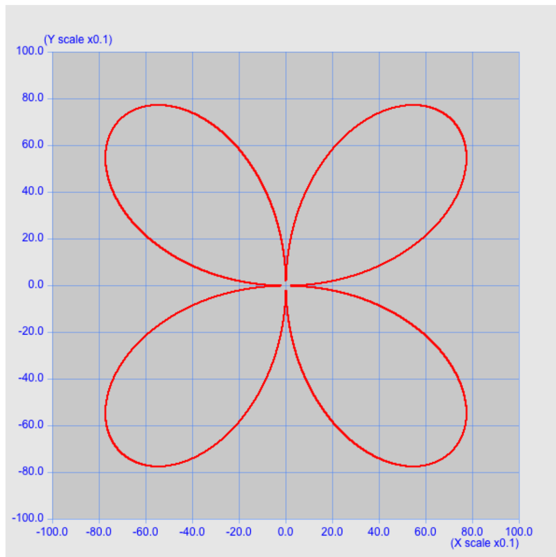


Figure 1: Lucky's gift to the class

(a) Find the derivative of Lucky's curve.

Solution: First we need to take the derivative with respect to x on both sides. Even though we have the RHS = 0, it is still good practice.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2+y^2\right)^3 - 404x^2y^2\frac{1}{10000}\right) = \frac{\mathrm{d}}{\mathrm{d}x}0$$

By recognizing the sum/difference rule of derivatives (i.e. $(f \pm g)' = f' \pm g'$), we uncover that we will need to use the chain rule for the first term and the product rule for the second term, highlighted in our typical color scheme.

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}x}\boxed{\left(\boxed{x^2+y^2}\right)^3} - \frac{\mathrm{d}}{\mathrm{d}x}404\boxed{x^2}\boxed{y^2} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{10000} \\
&= 3\left(\boxed{x^2+y^2}\right)^2 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\boxed{x^2+y^2}\right) - 404\left(x^2\frac{\mathrm{d}}{\mathrm{d}x}\boxed{\boxed{y}^2} + y^2(2x)\right) - 0 \\
&= 3\left(x^2+y^2\right)^2 \cdot \left(2x + \frac{\mathrm{d}}{\mathrm{d}x}\boxed{\boxed{y}^2}\right) - 404x^2\left(2\boxed{y}\frac{\mathrm{d}}{\mathrm{d}x}\boxed{y}\right) - 808xy^2 \\
&= 3\left(x^2+y^2\right)^2 \cdot \left(2x + 2\boxed{y}\frac{\mathrm{d}}{\mathrm{d}x}\boxed{y}\right) - 808x^2y\frac{\mathrm{d}y}{\mathrm{d}x} - 808xy^2 \\
&= 3\left(x^2+y^2\right)^2 \cdot \left(2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 808x^2y\frac{\mathrm{d}y}{\mathrm{d}x} - 808xy^2
\end{aligned}$$

Now that we have taken the derivative, this is the point where we would plug in our x, y values to make the algebra easier. However, we were asked to find the derivative for all x, y, so we must rearrange to isolate $\frac{\mathrm{d}y}{\mathrm{d}x}$. We first move all the terms that involve $\frac{\mathrm{d}y}{\mathrm{d}x}$ on one side, and everything else to the other side (recall, that the whole derivative equals zero.)

$$\begin{aligned}
-3(x^2+y^2)^2 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 808x^2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 3\left(x^2+y^2\right)^2 2x - 808xy^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(-6y(x^2+y^2)^2 + 808x^2y\right) &= 6x\left(x^2+y^2\right)^2 - 808xy^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6x\left(x^2+y^2\right)^2 - 808xy^2}{-6y(x^2+y^2)^2 + 808x^2y}
\end{aligned}$$

2. **Implicit differentiation.** The psychadelic 8-ball family of functions is described by

$$\frac{y^2}{2} + \cos\left(x^2+y^2\right) - \frac{a}{2} = 0, \text{ for } a = [1, 2, \cdots, 15].$$

The whole family is visualized in Fig 2(a), where we are going to be looking specifically at when $a = 7$, visualized in Fig 2(b).

(a) (3 points) What is the slope of the tangent line at $(x, y) = \left(-\sqrt{\frac{7\pi}{2} - 7}, \sqrt{7}\right) \approx (-1.99889, 2.645751)$

A. Positive B. Negative C. Zero D. Need More Information

(b) (3 points) What is the slope of the tangent line at $(x, y) = \left(-\sqrt{\frac{7\pi}{2} - 7}, \sqrt{7}\right) \approx (-1.99889, 2.645751)$

A. > 1 **B.** < 1 C. Zero D. Undefined

(c) (3 points) What techniques could we use to take the derivative?

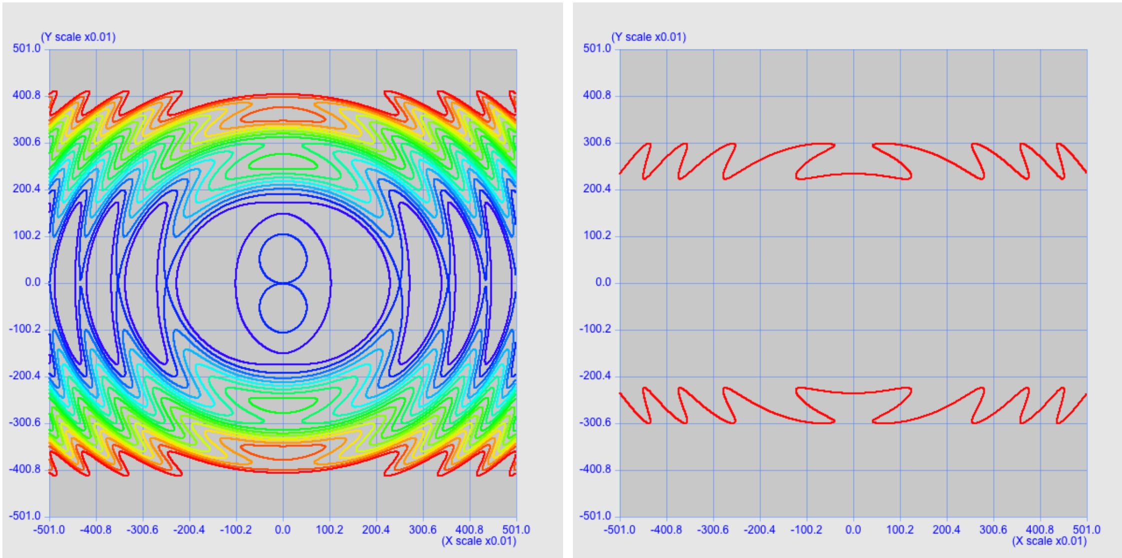


Figure 2: Psychodelic 8-ball family of functions (left), The one specific function when $a = 7$ (right)

- ☐ We only use the power rule every time we see x^2 or y^2
- ☐ We rewrite the function in terms of $y(x)$.
- $\surd$ **We use implicit differentiation**
- ☐ We use the product rule on cos multiplied by $(x^2 - y^2)$.

(d) (11 points) Using Implicit differentiation, find the equation of a tangent line of

$$\frac{y^2}{2} + \cos\left(x^2 + y^2\right) - \frac{7}{2} = 0 \text{ at } \left(-\sqrt{\frac{7\pi}{2} - 7}, \sqrt{7}\right)$$

Solution: First we start by taking the deriviative with respect to x of both sides, and using the sum/difference rule of derivatives.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y^2}{2}+\cos\left(x^2+y^2\right)-\frac{7}{2}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\frac{\boxed{\boxed{y}^2}}{2}+\frac{\mathrm{d}}{\mathrm{d}x}\boxed{\cos\left(\boxed{x^2+y^2}\right)}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{7}{2} &= 0 \\
\frac{2}{2}\boxed{y}\frac{\mathrm{d}}{\mathrm{d}x}\boxed{y}-\sin\left(\boxed{x^2+y^2}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\boxed{x^2+y^2}-0 &= 0 \\
y\frac{\mathrm{d}y}{\mathrm{d}x}-\sin\left(x^2+y^2\right)\left(2x+\frac{\mathrm{d}}{\mathrm{d}x}\boxed{\boxed{y}^2}\right) &= 0 \\
y\frac{\mathrm{d}y}{\mathrm{d}x}-\sin\left(x^2+y^2\right)\left(2x+2y\frac{\mathrm{d}}{\mathrm{d}x}\boxed{y}\right) &= 0 \\
y\frac{\mathrm{d}y}{\mathrm{d}x}-\sin\left(x^2+y^2\right)\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 0.
\end{aligned}$$

Since we have points to plug in, we do that now.

$$\begin{aligned}
\left(\sqrt{7}\right)\frac{\mathrm{d}y}{\mathrm{d}x}-\sin\left(\left(-\sqrt{\frac{7\pi}{2}-7}\right)^2+\left(\sqrt{7}\right)^2\right)\left(2\left(-\sqrt{\frac{7\pi}{2}-7}\right)+2\left(\sqrt{7}\right)\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 0 \\
\sqrt{7}\frac{\mathrm{d}y}{\mathrm{d}x}-\sin\left(\left(\frac{7\pi}{2}-7\right)+(7)\right)\left(2\left(-\sqrt{\frac{7\pi}{2}-7}\right)+2\left(\sqrt{7}\right)\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 0 \\
\sqrt{7}\frac{\mathrm{d}y}{\mathrm{d}x}-\sin\left(\frac{7\pi}{2}\right)\left(2\left(-\sqrt{\frac{7\pi}{2}-7}\right)+2\left(\sqrt{7}\right)\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 0 \\
\sqrt{7}\frac{\mathrm{d}y}{\mathrm{d}x}-2\left(\sqrt{\frac{7\pi}{2}-7}\right)+2\sqrt{7}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
2\sqrt{\frac{7\pi}{2}-7} &= 3\sqrt{7}\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{2\sqrt{\frac{7\pi}{2}-7}}{3\sqrt{7}} &= \frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{2}{3}\sqrt{\frac{\pi}{2}-1} &= \frac{\mathrm{d}y}{\mathrm{d}x} \\
0.50367 &\approx \frac{\mathrm{d}y}{\mathrm{d}x}
\end{aligned}$$

Lastly, we note that this is a positive number that is less than 1, so it fits in with the visual information we have already applied.

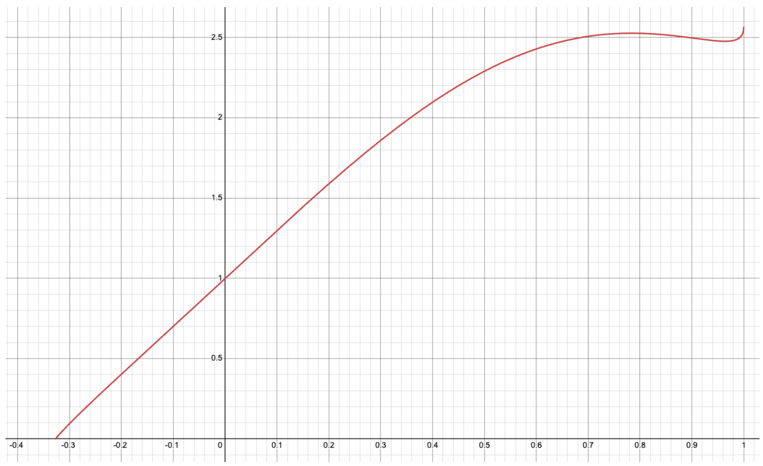
3. **Logarithmic Differentiation** The end goal of this is to take the derivative of

$$f(x) = (1 + 3\arcsin(x))^{\cos(x)}$$

using the logarithmic differentiation technique outlined below.

(a) (2 points) Is there ever a point where $(1 + 3\arcsin(x))^{\cos(x)} \leq 0$?

Solution: No. First, we know that the domain of $\arcsin(x) = [-1, 1]$ and the domain of $\cos(x)$ is all real numbers. However, once we get something that is less than 0 to a negative power, we could end up with something as nonsensical as $\frac{1}{\sqrt{-2}}$. This means the base of the exponent cannot be negative, which means that the domain of this function is $x = [-\sin\left(\frac{1}{3}\right), 1]$.
If you don't believe the above argument, you can see for yourself



(b) (2 points) What is $\ln f(x)$?

Solution: Using the property of logarithms, that $\log_b(x^y) = y\log_b(x)$, we are able to see that

$$\begin{aligned} f(x) &= (1 + 3\arcsin(x))^{\cos(x)} \\ \ln f(x) &= \ln\left((1 + 3\arcsin(x))^{\cos(x)}\right) \\ &= \cos(x)\ln\left((1 + 3\arcsin(x))\right) \end{aligned}$$

(c) (2 points) What is $\frac{\mathrm{d}}{\mathrm{d}x}\ln f(x)$ using the chain rule?

Solution: This is a chain rule situation, so we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\boxed{\ln\left(\boxed{f(x)}\right)}\right) &= \frac{1}{\boxed{f(x)}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\boxed{f(x)} \\ &= \frac{f'(x)}{f(x)} \end{aligned}$$

(d) (9 points) What is $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$?

Solution: Combining the work from (b) and (c) above, we will use the *logarithmic derivative*. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(f(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left((1+3\arcsin(x))^{\cos(x)}\right)\right)\\
\frac{f'(x)}{f(x)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\boxed{\cos(x)}\boxed{\ln\left(1+3\arcsin(x)\right)}\right)\\
&= \cos(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\boxed{\ln\left(\boxed{1+3\arcsin(x)}\right)}\right) + \ln\left(1+3\arcsin(x)\right)\frac{\mathrm{d}}{\mathrm{d}x}\cos(x)\\
&= \cos(x)\frac{1}{1+3\arcsin(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\boxed{1+3\arcsin(x)}\right) - \ln\left(1+3\arcsin(x)\right)\sin(x)\\
&= \cos(x)\frac{1}{1+3\arcsin(x)}\left(\frac{-3}{\sqrt{1-x^2}}\right) - \ln\left(1+3\arcsin(x)\right)\sin(x)\\
f'(x) &= f\left(\cos(x)\frac{-3}{(1+3\arcsin(x))\sqrt{1-x^2}} - \ln\left(1+3\arcsin(x)\right)\sin(x)\right)
\end{aligned}$$

Remembering what $f(x)$ is, we end up with a final answer of

$$f'(x) = (1+3\arcsin(x))^{\cos(x)}\left(\cos(x)\frac{-3}{(1+3\arcsin(x))\sqrt{1-x^2}} - \ln\left(1+3\arcsin(x)\right)\sin(x)\right)$$

4. **Logarithmic Differentiation** The end goal of this is to find the equation of the tangent line at $(x, y) = (1, 1)$ of the function

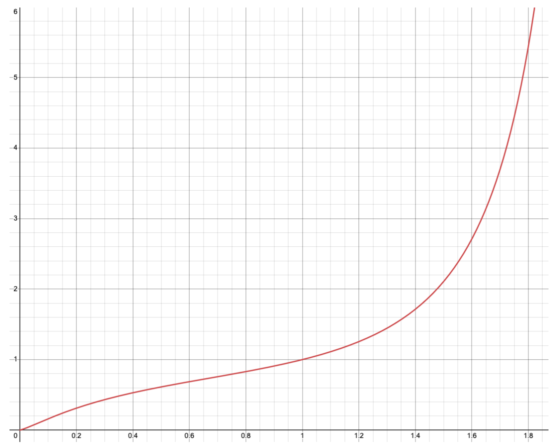
$$g(x) = x^{x^x}$$

using the logarithmic differentiation technique outlined below.

(a) (2 points) Is there ever a point where $x^{x^x} \leq 0$?

Solution: No. Assuming that the domain of $f(x)$ are all real x values that are greater than 0, then we will always have the range is greater than 0.

If you don't believe the above argument, you can see for yourself



(b) (3 points) What is $\ln g(x)$?

Solution: Using the property of logarithms, that $\log_b(x^y) = y\log_b(x)$, we are able to see that

$$\begin{aligned} g(x) &= x^{x^x} \\ \ln g(x) &= \ln\left(x^{x^x}\right) \\ &= x^x \ln(x) \end{aligned}$$

(c) (2 points) What is $\frac{\mathrm{d}}{\mathrm{d}x} \ln g(x)$ using the chain rule?

Solution: This is a chain rule situation, so we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\boxed{\ln\left(\boxed{g(x)}\right)}\right) &= \frac{1}{\boxed{g(x)}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\boxed{g(x)} \\ &= \frac{g'(x)}{g(x)} \end{aligned}$$

(d) (10 points) What is $\frac{\mathrm{d}}{\mathrm{d}x} g(x)$?

Solution: Combining the results from (b) and (c). we see that

$$\begin{aligned} \frac{g'(x)}{g(x)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\boxed{x^x}\,\boxed{\ln(x)}\right) \\ &= x^x \frac{\mathrm{d}}{\mathrm{d}x} \ln x + \ln(x) \frac{\mathrm{d}}{\mathrm{d}x} x^x \\ &= x^x \frac{1}{x} + \ln(x) \frac{\mathrm{d}}{\mathrm{d}x} x^x \end{aligned}$$

Since in class we already computed that $\frac{\mathrm{d}}{\mathrm{d}x}x^x = x^x(1+\ln(x))$, we can directly substitute that in here, to get

$$\begin{aligned}
\frac{g'(x)}{g(x)} &= x^x\frac{1}{x} + \ln(x)\left(x^x(1+\ln(x))\right) \\
&= x^{x-1} + \ln(x)\left(x^x + x^x\ln(x)\right) \\
g'(x) &= g\left(x^{x-1} + \ln(x)\left(x^x + x^x\ln(x)\right)\right) \\
g'(x) &= x^{x^x}\left(x^{x-1} + \ln(x)\left(x^x + x^x\ln(x)\right)\right)
\end{aligned}$$

(e) (3 points) Using point slope form, what is the equation for the tangent line at (1,1).

Solution: Our answer will be the equation of the tangent line in point-slope form, so since we already have the point $(x, y) = (1, 1)$, we just need $g'(1)$, but we already have $g'(x)$, so we just plug 1 in

$$\begin{aligned}
g'(1) &= 1^{1^1}\left(1^{1-1} + \ln(1)\left(1^1 + 1^1\ln(1)\right)\right) \\
&= 1\left(1^0 + 0\,(1+0)\right) \\
&= 1
\end{aligned}$$

Now that we have the point and the slope, we have the equation of the tangent line:

$$\begin{aligned}
y - 1 &= 1(x-1) \\
y(x) &= x
\end{aligned}$$

5. **Optimization with Implicit Differentiation** The City of St. Louis, MO, is having a housing crisis, and needs to build more houses. The mayor, Cara Spencer, has approved the construction of a new housing complex under the St. Louis Arch, which can be modelled as a conic section equation,

$$f(x, y(x)) = 3x^2 + 350y - 215250 = 0,$$

where y is the height in feet, x is the horizontal length of the arch in feet, and $x = 0$ is right under the cornerstone of the Arch.[1] Mayor Cara Spencer has hired your consulting firm to find the maximum area of the facade of the housing complex.

Pictorially, we are trying to find the maximum area of the shaded purple region.

[1] The St. Louis Arch is **NOT** actually a parabola. In fact the equation is related to $\cosh(x) = \frac{e^{ax}+e^{-ax}}{2}$, where a is a constant.

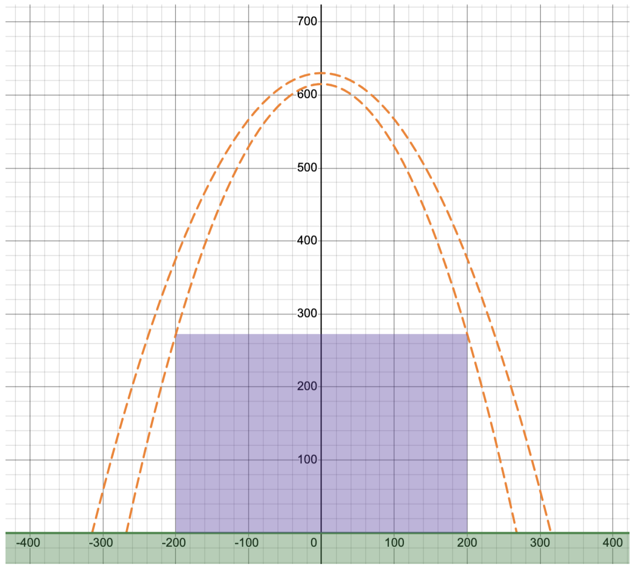


Figure 3: The schematic of the St. Louis Arch, and the purple building you want to maximize the area.

(a) (3 points) Conceptually explain why the maximum area of the facade has to be touching the St. Louis Arch.

> **Solution:** Let's assume that we have the length of the facade is 400 ft, from -200 to 200 ft. However, if we only build the building to be 200 ft, then we can still build higher under the St. Louis Arch without making the base wider. This means since we want to maximize the area, we should build more height until we touch the Arch.

(b) (7 points) Find the maximum area of the facade. *Hint: from above, we are looking for a point a such that* $af(a)$*=length×height is maximal.*

Solution: Since we know that we can never achieve the maximum area if the building doesn't touch the St. Louis Arch, then we are trying to find a point a, which is the distance from the center of the arch on the ground, where $f(a)$ is the height. Since we want to find the area of the facade, we are looking for length $\times$ height $= 2a \times f(a)$, which is maximal. Notice the 2 comes from the vertical symmetry of the problem.

First, we need to find the derivative of this surface area function (recall: $xy(x) = A$). Since we have an implicit function for the height, we can multiply it by the x to get

$$\begin{aligned} x\left(3x^2 + 350y - 215250 = 0\right) = \implies & 3x^3 + 350xy - 215250x = 0 \\ \implies & 3x^3 + 350A(x) - 215250x = 0 \\ \overset{\frac{\mathrm{d}}{\mathrm{d}x}}{\implies} & 9x^2 + 350\frac{\mathrm{d}A}{\mathrm{d}x} - 215250 = 0 \\ \implies & \frac{\mathrm{d}A}{\mathrm{d}x} = \frac{-9}{350}x^2 + 615. \end{aligned}$$

Since this derivative is defined everywhere, we don't have to worry about the points where $\frac{\mathrm{d}A}{\mathrm{d}x}$ does not exist. So this means all the critical points come from when the derivative is zero.

$$\begin{aligned} \frac{-9}{350}x^2 + 615 &= \frac{\mathrm{d}A}{\mathrm{d}x} \\ &= 0 \\ x^2 &= -615 \cdot \frac{350}{-9} \\ &= 23916.67 \\ x &= 154.6 \text{ ft} \end{aligned}$$

We also see that $x = -154.6$ ft would be a solution. In fact this is the same solution we found, but considering the left side, not the right side. Since we are looking for positive x, we have found our only critical point. We are going to use the 3 other methods (not including graphing, which we already did) to solve this problem.

First Derivative Method We are going to consider points less than and greater then our only critical point.

x	0	154.6	200
$A(x)$	$\nearrow$	$\rightarrow$	$\searrow$
$A'(x)$	615	0	- 413.57

By the First Derivative Test, we have that $x = 154.6$ is the local max. Since this is the only critical point, this means that $x = 154.6$ is a global max as well. This means that $y(154.6) = 210.733$, so the total area of the facade is $2xy(x) = 2(154.6)y(154.6) = 65,179.777$ sq.ft.

Second Derivative Test First, we need to take the derivative of this function again

$$A''(x) = \frac{-18}{350}x$$
$$A''(154.6) = \frac{-18}{350}(154.6) < 0$$

By the Second Derivative Test, we have that the second derivative is negative at the critical point, which means that the function is concave down at $x = 154.6$, so it is the local max. Since this is the only critical point, this means that $x = 154.6$ is a global max as well. This means that $y(154.6) = 210.733$, so the total area of the facade is $2xy(x) = 2(154.6)y(154.6) = 65,179.777$ sq.ft.

Closed Interval Method Since we found one critical point, we need to find the domain of this function. Clearly x, which is the length from the center of the Arch has to be positive, so if $x = 0, y(0) = 615$. Additionally, x has to be less than the width of the St. Louis Arch, which is 268 ft. Plugging this into the table below:

x	0	154.6	268
$A(x)$	0 sq.ft.	65,179.777 sq.ft.	0 sq.ft.

we see that $x = 154.6$ ft yields the largest area of the facade.