Emory University

Review for Exam # 2

MA-111 Calculus I

Date: October 24, 2025

Instructor: Prof. Mitchell Scott

Student ID ____________ Name: ________________________

Please read the following instructions carefully.

- You have 90 minutes to complete this exam. This question booklet contains 5 questions, 8 pages (including the cover) for the total of 75 points/marks. Check to see if any pages are missing. DO NOT scribble or do rough work or make any stray marks on it. Use separate sheet for rough work.
- All the questions are compulsory and all the notations have their usual meaning. You can use standard results directly by writing its statement appropriately. **Read the instructions for individual questions carefully** before answering the questions.
- More instructions here.

Question	Points	Score
1	10	
2	20	
3	15	
4	20	
5	10	
Total:	75	

1. (10 points) **Implicit differentiation.** Lucky the Leprechaun wants to give you a nice token, as he knows some people might be stressed. He concocts the following curve

$$\left(x^2+y^2\right)^3 - 404x^2y^2 - \frac{1}{10000} = 0$$ 25

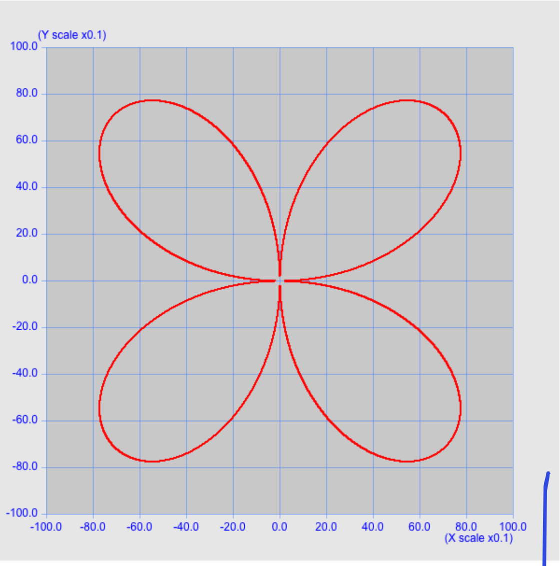


Figure 1: Lucky's gift to the class

$y(x) = y$

$\frac{d}{dx}\boxed{\boxed{y}^2} = 2y\frac{dy}{dx}$

(a) Find the derivative of Lucky's curve.

$$\frac{d}{dx}\boxed{(\boxed{x^2+y^2})^3} - \frac{d}{dx}404\boxed{x^2}\boxed{y^2} - \frac{d}{dx}\frac{1}{10000} = \frac{d}{dx}0$$

$$3(x^2+y^2)^2 \cdot \frac{d}{dx}(x^2+y^2) - 404\left(x^2\frac{d}{dx}y^2 + y^2(2x)\right) - 0 = 0$$

$$3(x^2+y^2)^2\left(2x + 2y\frac{dy}{dx}\right) - 404\left(x^2\, 2y\frac{dy}{dx} + 2xy^2\right) = 0$$

$$-3(x^2+y^2)^2(2y)\frac{dy}{dx} + 808x^2y\frac{dy}{dx} = 3(x^2+y^2)^2 2x - 808xy^2$$

$$\frac{dy}{dx}\left(-6y(x^2+y^2)^2 + 808x^2y\right) = 6x(x^2+y^2)^2 - 808xy^2$$

$$\boxed{\frac{dy}{dx} = \frac{6x(x^2+y^2)^2 - 808xy^2}{-6y(x^2+y^2)^2 + 808x^2y}}$$

$\cdot\frac{-1}{-1}$

2. **Implicit differentiation.** The psychadelic 8-ball family of functions is described by

$$\frac{y^2}{2} + \cos\left(x^2 + y^2\right) - \frac{a}{2} = 0, \text{ for } a = [1, 2, \cdots, 15].$$

The whole family is visualized in Fig 2(a), where we are going to be looking specifically at when $a = 7$, visualized in Fig 2(b).

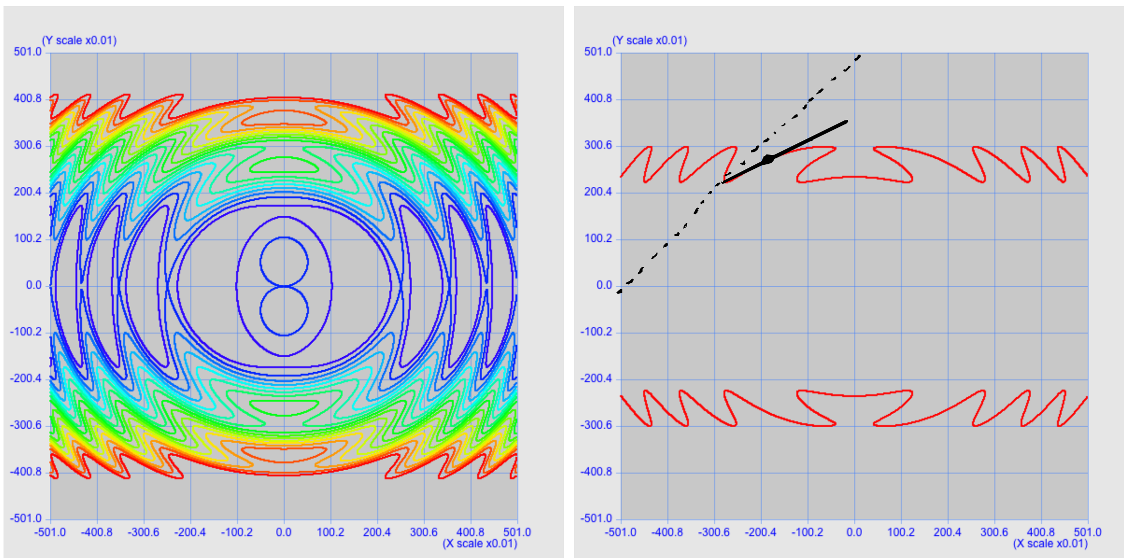


Figure 2: Psychodelic 8-ball family of functions (left), The one specific function when $a = 7$ (right)

(a) (3 points) What is the slope of the tangent line at $(x, y) = \left(-\sqrt{\frac{7\pi}{2} - 7}, \sqrt{7}\right) \approx (-1.99889, 2.645751)$

A. Positive B. Negative C. Zero D. Need More Information

(b) (3 points) What is the slope of the tangent line at $(x, y) = \left(-\sqrt{\frac{7\pi}{2} - 7}, \sqrt{7}\right) \approx (-1.99889, 2.645751)$

A. > 1 B. < 1 C. Zero D. Undefined

(c) (3 points) What techniques could we use to take the derivative?

☒ We only use the power rule every time we see x^2 or y^2
☒ We rewrite the function in terms of $y(x)$.
☑ We use implicit differentiation
☒ We use the product rule on cos multiplied by $(x^2 + y^2)$.

$y = y(x)$

(d) (11 points) Using Implicit differentiation, find the equation of a tangent line of

$$\frac{d}{dx}\left(\frac{y^2}{2} + \cos\left(x^2+y^2\right) - \frac{7}{2}\right) = \frac{d}{dx}0 \text{ at } \left(-\sqrt{\frac{7\pi}{2}-7}, \sqrt{7}\right)$$

$$\frac{d}{dx}\frac{y^2}{2} = \frac{2(2y\frac{dy}{dx}) - y^2 0}{4} = \frac{4y\frac{dy}{dx}}{4} = \frac{dy}{dx}$$

$$\frac{d}{dx}\left(\boxed{\frac{\boxed{y^2}}{2}}\right) + \frac{d}{dx}\boxed{\cos\left(\boxed{x^2+y^2}\right)} - \frac{d}{dx}\frac{7}{2} = 0$$

$$\frac{1}{2}2y\frac{dy}{dx} + -\sin(x^2+y^2)\cdot\left(2x + 2y\frac{dy}{dx}\right) - 0 = 0$$

$x = -\sqrt{\frac{7\pi}{2} - 7}$ $y = \sqrt{7}$

$$\sqrt{7}\frac{dy}{dx} - \sin\left(\left(-\sqrt{\frac{7\pi}{2}-7}\right)^2 + \left(\sqrt{7}\right)^2\right)\cdot\left(-2\sqrt{\frac{7\pi}{2}-7} + 2\sqrt{7}\frac{dy}{dx}\right) = 0$$

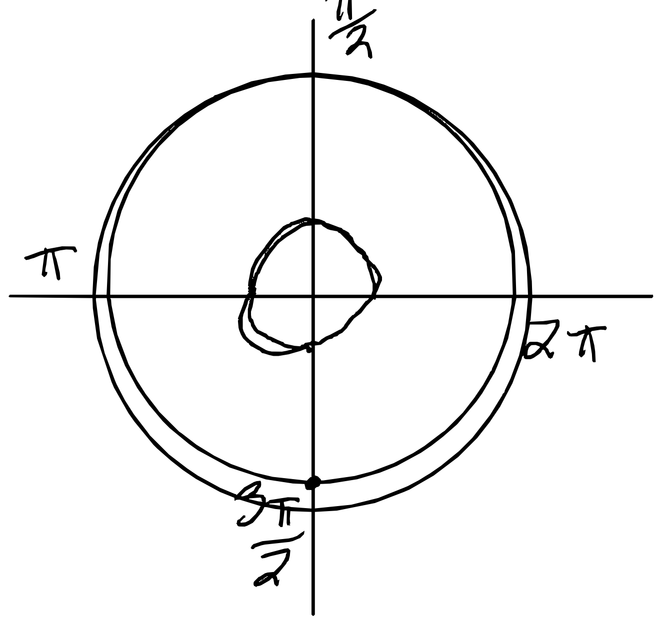
$$\sqrt{7}\frac{dy}{dx} - \sin\left(\frac{7\pi}{2} - 7 + 7\right)\cdot\left(-2\sqrt{\frac{7\pi}{2}-7} + 2\sqrt{7}\frac{dy}{dx}\right) = 0$$

$$1\sqrt{7}\frac{dy}{dx} - 2\sqrt{\frac{7\pi}{2}-7} + 2\sqrt{7}\frac{dy}{dx} = 0$$

$$3\sqrt{7}\frac{dy}{dx} = 2\sqrt{\frac{7\pi}{2}-7}$$

$$\frac{dy}{dx} = \frac{2\sqrt{\frac{7\pi}{2}-7}}{3\sqrt{7}}$$

$$\approx 0.503$$



3. **Logarithmic Differentiation** The end goal of this is to take the derivative of

$$f(x) = (1 + 3\arcsin(x))^{\cos(x)}$$

$\ln(xy) = \ln x + \ln y$

using the logarithmic differentiation technique outlined below.

(a) (2 points) Is there ever a point where $(1 + 3\arcsin(x))^{\cos(x)} \leq 0$?

no!

(b) (2 points) What is $\ln f(x)$?

$$\frac{d}{dx} \quad \ln f(x) = \ln\left(1+3\arcsin(x)\right)^{\cos(x)} = \cos(x)\ln\left(1+3\arcsin(x)\right)$$

(c) (2 points) What is $\frac{\mathrm{d}}{\mathrm{d}x} \ln f(x)$ using the chain rule?

$$\frac{d}{dx}\boxed{\ln\left(\boxed{f(x)}\right)} = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

(d) (9 points) What is $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$?

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\left(\boxed{\cos(x)\,\boxed{\ln\left(\boxed{1+3\arcsin(x)}\right)}}\right)$$

$$= \cos(x)\frac{1}{1+3\arcsin(x)} \cdot 3\left(\frac{+1}{\sqrt{1-x^2}}\right) + \ln(1+3\arcsin(x))(-\sin(x))$$

$$f'(x) = f\left(\frac{3\cos}{(1+3\arcsin(x))(\sqrt{1-x^2})} - \ln(1+3\arcsin(x))\sin(x)\right)$$

$$= (1+3\arcsin(x))^{\cos x}\left(\frac{3\cos}{(1+3\arcsin(x))(\sqrt{1-x^2})} - \ln(1+3\arcsin(x))\sin(x)\right)$$

4. **Logarithmic Differentiation** The end goal of this is to find the equation of the tangent line at $(x, y) = (1, 1)$ of the function

$$g(x) = [redacted] \quad x^{\sqrt{x}}$$

$\ln x^{x^x} = x^x \ln x$

$x^x \frac{1}{x} + \ln x \frac{d}{dx} x^x$

using the logarithmic differentiation technique outlined below.

(a) (2 points) Is there ever a point where $x^{x^x} \leq 0$?

(b) (3 points) What is $\ln g(x)$?

$\ln g(x) = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$

(c) (2 points) What is $\frac{\mathrm{d}}{\mathrm{d}x} \ln g(x)$ using the chain rule?

$\frac{g'(x)}{g(x)} \qquad \frac{1}{g(x)} \cdot g'(x)$

(d) (10 points) What is $\frac{\mathrm{d}}{\mathrm{d}x} g(x)$?

$\sqrt{x} = x^{1/2}$

$\frac{g'}{g} = \frac{d}{dx} \boxed{\sqrt{x} \mid \ln x}$

$\frac{x^{1/2}}{x'} = x^{\frac{1}{2}-1} = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$

$x^{1/2} \cdot \frac{1}{x'} + \ln x \, \frac{1}{2} x^{-1/2}$

$\frac{2 \cdot 1}{2\sqrt{x}} + \frac{\ln x}{2\sqrt{x}} \Rightarrow g' = x^{\sqrt{x}} \left(\frac{2 + \ln x}{2\sqrt{x}} \right)$

(e) (3 points) Using point slope form, what is the equation for the tangent line at (1,1).

$g'(1) = 1^1 \left(\frac{2+0}{2} \right) = 1 \qquad y(x) = 1(x-1) + 1$

$y = x$

5. **Optimization with Implicit Differentiation** The City of St. Louis, MO, is having a housing crisis, and needs to build more houses. The mayor, Cara Spencer, has approved the construction of a new housing complex under the St. Louis Arch, which can be modelled as a conic section equation,

$y = -\frac{3x^2}{350} + 615$

[redacted] $= 3x^2 + 350y - 215250 = 0,$

where y is the height in feet, x is the horizontal length of the arch in feet, and $x = 0$ is right under the cornerstone of the Arch.[1] Mayor Cara Spencer has hired your consulting firm to find the maximum area of the facade of the housing complex.

Pictorially, we are trying to find the maximum area of the shaded purple region.

$2af(a)$

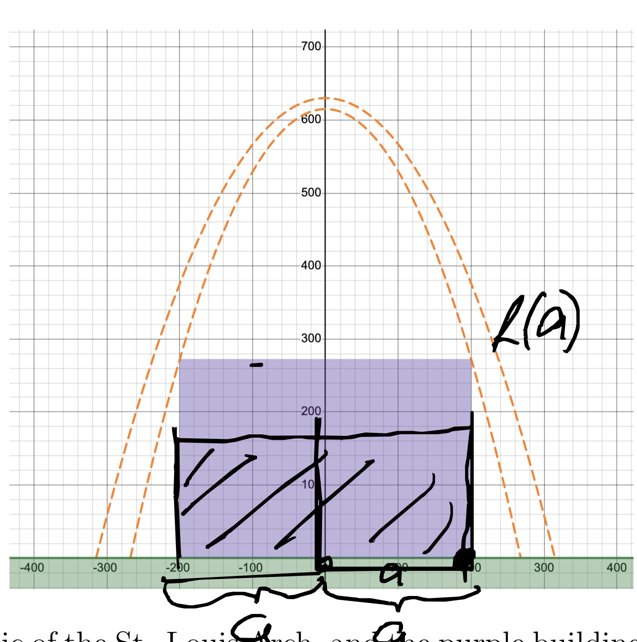


Figure 3: The schematic of the St. Louis Arch, and the purple building you want to maximize the area.

(a) (3 points) Conceptually explain why the maximum area of the facade has to be touching the St. Louis Arch.

[1]The St. Louis Arch is **NOT** actually a parabola. In fact the equation is related to $\cosh(x) = \frac{e^{ax}+e^{-ax}}{2}$, where a is a constant.

(b) (7 points) Find the maximum area of the facade. *Hint: from above, we are looking for a point a such that $af(a)$=length×height is maximal.*

$$\text{Area} = 2af(a) = 2a\left(\frac{-3a^2}{350} + 615\right)$$

$$= 2\left(\frac{-3a^3}{350} + 615a\right) = \frac{-6a^3}{350} + 1230a$$

$$A'(a) = \frac{-18a^2}{350} + 1230$$

$A'(a)$ DNE

$A'(a) = 0$

$$0 = \frac{-18a^2}{350} + 1230$$

$$\frac{18a^2}{350} = 1230 \Rightarrow a^2 = 23917$$

$$a = \pm 154.6 \text{ ft}$$

$$\frac{d}{dx} = \frac{d}{dx}\frac{-18a^2}{350}$$

$$= \frac{d}{dx}\frac{-18}{350}a^2$$

$$= \frac{-18}{350}\frac{d}{dx}a^2$$

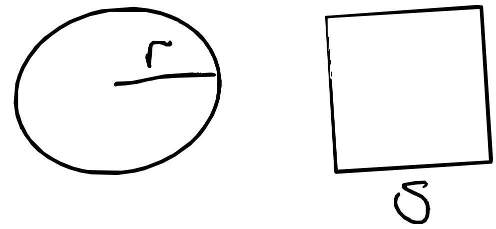
$$= \frac{-36}{350}a < 0$$

$$A = 2(154.6)\left(\frac{154.6^2 \cdot (-3)}{350} + 615\right)$$

$$= 65,179.8 \text{ sqft}$$

Goal: maximize area (possibly of circle/square)

$$A = \pi r^2 + s^2$$



$$2 = 2\pi r + 4s \qquad 2 = \frac{2\pi \cdot 1}{4+\pi} + 4s$$

$$\frac{2 - 2\pi r}{4} = \frac{4s}{4} \Rightarrow s = \frac{1}{2} - \frac{1}{2}\pi r \qquad 8 + 2\pi = 2\pi + 4s$$

$$A(r) = \pi r^2 + \left(\frac{1}{2} - \frac{1}{2}\pi r\right)^2$$

$$A'(r) = 2\pi r + 2\left(\frac{1}{2} - \frac{1}{2}\pi r\right)\cdot\left(-\frac{\pi}{2}\right)$$

$$= 2\pi r + (1 - \pi r)\cdot -\frac{\pi}{2} = 2\pi r - \frac{\pi}{2} + \frac{\pi^2}{2} r$$

$$= \left(2\pi + \frac{\pi^2}{2}\right) r - \frac{\pi}{2}$$

$$0 = \left(2\pi + \frac{\pi^2}{2}\right) r - \frac{\pi}{2}$$

$$r = \frac{\pi}{2\left(2\pi + \frac{\pi^2}{2}\right)} = \frac{\pi}{4\pi + \pi^2} = \frac{\pi}{\pi^2 + 4\pi} = \frac{\pi}{\pi(\pi+4)}$$

$$r = \frac{1}{4+\pi}$$

$$A = \pi r^2 + \left(\frac{1}{2} - \frac{1}{2}\pi r\right)^2$$

$$\frac{\pi}{(4+\pi)^2} + \left(\frac{1 - \pi\left(\frac{1}{4+\pi}\right)}{2}\right)^2$$