MATH 190: MATHEMATICS OF SUSTAINABILITY
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Fall 2024
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Solutions to Homework # 5

Markov Chains: Definitions and Examples

SP §1.1

1. **A fair coin is tossed repeatedly with results $Y_0, Y_1, Y_2, \ldots$ that are 0 or 1 with probability 1/2 each. For $n \geqslant 1$ let $X_n = Y_n + Y_{n-1}$ be the number of 1's in the $(n-1)$th and nth tosses. Is X_n a Markov chain?**

   Solution. First we need to show that $P(X_{n+1} = j | X_n = i, X_{n-1} = i'), \forall i', i, j$ for X_n to be a Markov Chain. Let $j = 2,\ i = 1,\ i' = 0$. This results in $P(X_{n+1} = j | X_n = i, X_{n-1} = i') = 0.5$ because the only way $j = 2$ is if $i = 1, i' = 1$ and all other rolls were 0. Now let's assume that the $X_{n-1} = 2$, which can happen if a zero was rolled at the nth time, and a 1 is rolled at the next time. But this means that $X_{n+1} \neq 2$, so $P(X_{n+1} = j | X_n = i, X_{n-1} = i') \neq P(X_{n+1} = j | X_n = i) \forall i', i, j$, so this proves this isn't a Markov Chain.

2. **Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n.**

   Solution. The first step in computing the probability transition matrix, M, for this process is knowing that the number of white balls in the left bucket can only consist of $0, 1, 2, 3, 4, 5$. There is also a nice symmetry in this problem because the probability of moving from n white balls to $n + 1$ white balls is also the probability from moving $5 - n$ black balls to $4 - n$ black balls. Let's start with calculating the probability of there being zero white balls in the left bucket and go to there being 2 white balls in the left bucket; the rest is just symmetric. If there are zero white balls in the left bucket, that means there are 5 white balls in the right bucket, so you are guaranteed to then have one white ball in the left bucket. Since this is a Markov chain, and all rows must sum to one, all other entries in the first row equals zero.

   If there is one white ball in the left bucket, there is a possibility after the transition to have 0, 1, or 2 white balls in the left bucket. The entries in 3,4, and 5 have a zero probability. Assuming the two ball choices are independent events, if there is one white ball in the left bucket, there is a $\frac{1}{5}$ the white ball is grabbed in the left bucket and a $\frac{1}{5}$ chance a black ball is grabbed from the right bucket meaning, $p(1, 0) = \frac{1}{25}$. If there is one white ball in the left bucket, and you need to stay at one white ball, there can be two routes– switching white balls or switching black balls. To switch white balls, the probability of that is $\frac{1}{5} \cdot \frac{4}{5}$ or $\frac{4}{25}$. To switch black balls the probability is $\frac{4}{5} \cdot \frac{1}{5}$,or $\frac{4}{25}$. Adding these together is $p(1,1) = \frac{8}{25}$. Because this is a Markov chain all rows must sum to one so $p(1,2) = \frac{12}{25}$.

Lastly, if there are 2 white balls in the left bucket, basic probability calculations yield $p(2,1) = \frac{4}{25}$, $p(2,2) = \frac{12}{25}$, and $p(2,3) = \frac{9}{25}$. Using the symmetry that was discussed above the probability transition matrix, M, is below.

	0	1	2	3	4	5
0	0	1	0	0	0	0
1	1/25	8/25	16/25	0	0	0
2	0	4/25	12/25	9/25	0	0
3	0	0	9/25	12/25	4/25	0
4	0	0	0	16/25	8/25	1/25
5	0	0	0	0	1	0

Table 1: Probability Transition Matrix, M, for number of White Balls in the left bucket

3. **We repeated roll two four sided dice with numbers 1, 2, 3, and 4 on them. Let Y_k be the sum on the kth roll, $S_n = Y_1 + \cdots + Y_n$ be the total of the first n rolls, and $X_n = S_n(\mod 6)$. Find the transition probability for X_n**

Solution. The probability of rolling 2-four sided die has the following break down where the sides of the matrix are what appear on the subsequent dice with the sum (mod 6) in the actual matrix. This means that for example the possibility of having

	1	2	3	4
1	2	3	4	5
2	3	4	5	0
3	4	5	0	1
4	5	0	1	2

Table 2: Sum of the 2-four sided dice (mod 6)

the sum not change after rolling the dice is 3/16 (since there are three 0's in the matrix.) or $p(x,x) = 3/16$. Using the probability for each number in the table and traversing over all x values gives you the following probability transition matrix. A

	0	1	2	3	4	5
0	3/16	2/16	2/16	2/16	3/16	4/16
1	4/16	3/16	2/16	2/16	2/16	3/16
2	3/16	4/16	3/16	2/16	2/16	2/16
3	2/16	3/16	4/16	3/16	2/16	2/16
4	2/16	2/16	3/16	4/16	3/16	2/16
5	2/16	2/16	2/16	3/16	4/16	3/16

Table 3: Probability Transition Matrix, M, for the sum of the dice (mod 6)

good way to double check that this is the correct probability transition matrix is sum the columns and see if they sum to 1, which they all do.

SP §1.2

4. **The 1990 census showed that 36% of the households in the District of Columbia were homeowners while the remainder were renters. During the next decade 6% of the homeowners became renters and 12% of the renters became homeowners. What percentage were homeowners in 2000? in 2010?**

   Solution. First it is good to note that it is assumed that the 6% and 12% figures hold in the transition between the 2000 and 2010 US Census. The next step is to calculate the probability transition matrix. This is then multiplied with the

	H	R
H	0.94	0.06
R	0.12	0.88

Table 4: Matrix displaying the breakdown between homeowners (H) and renters (R) in Washington, D.C and the transition from one census to the next.

   matrix [0.36 0.64], which came from the problem statement, where the first column is homeowners and then renters. This gives you the matrix [0.4152 0.5848] for the 2000 US Census. Now multiplying the 2000 US Census matrix with the probability transition matrix which gives you the 2010 US census matrix of [0.460464 0.539536] with the columns of homeowners and renters respectively.

5. **Consider a gambler's ruin chain with $n = 4$. That is, if $1 \leqslant i \leqslant 3$, $p(i, i+1) = 0.4$, and $p(i, i-1) = 0.6$, but the endpoints are absorbing states: $p(0,0) = 1$ and $p(4,4) = 1$. Compute $p^3(1,4)$ and $p^3(1,0)$.**

   Solution. First, let's start with making M, the probability transition matrix for this Markov Chain. The probabilities after three steps can be calculated by cubing the

	0	1	2	3	4
0	1	0	0	0	0
1	0.6	0	0.4	0	0
2	0	0.6	0	0.4	0
3	0	0	0.6	0	0.4
4	0	0	0	0	1

Table 5: Probability Transition Matrix, M, for the Gambler's Ruin

   probability transition matrix, M, which is seen in Table 2, which implies that the probability after three bets, that the gambler has \$0 or \$4 ($p^3(1,0)$ and $p^3(1,4)$) is 0.744 and 0.064, respectively.

6. **A taxicab driver moves between the airport$A$ and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4.**

   (a) **Find the transition matrix for the chain.**

	0	1	2	3	4
0	1	0	0	0	0
1	0.744	0	0.192	0	0.064
2	0.360	0.288	0	0.192	0.160
3	0.216	0	0.288	0	0.496
4	0	0	0	0	1

Table 6: Matrix M after 3 consecutive bets

(b) **Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.**

Solution. Given the problem statement, the following probability transition matrix can be found. Squaring this matrix to get the two step probability transition matrix,

	A	B	C
A	0	0.5	0.5
B	0.75	0	0.25
C	0.75	0.25	0

Table 7: Matrix, M, showing the location of the taxi at either the airport (A), Hotel B, or Hotel C

and reading off the first row to get the probability starting from the airport is [0.75 0.125 0.125]. Then finding the three step probability transition matrix by cubing M, the probability that the cab is at Hotel B is 0.40625.

7. **Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$.**

   (a) **Compute its transition probability.**

   (b) **Compute the two-step transition probability.**

   (c) **What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday**

Solution. First let's start out by finding the probability transition matrix. Using what was given in the problem statement and knowing that this is a Markov chain, so the row probabilities must add up to 1, the probability transition matrix is trivial. Then squaring the matrix, M, to get the two step probability transition matrix. For part c, there are only two different ways that can compare Sunday and Monday to Wednesday– whether or not it rained on Tuesday. If it rained on Tuesday mathematically is calculated by SS by SR in the probability transition matrix, 0.3 times 0.6, or the probability that it rained if one of the two days before rained. Doing the same sort of logic assuming that it didn't rain on Tuesday gives us $0.3 \cdot 0.7 = 0.21$. Adding 0.18 and 0.21 gives us the answer of 0.39.

	RR	RS	SR	SS
RR	0.6	0.4	0	0
RS	0	0	0.6	0.4
SR	0.6	0.4	0	0
SS	0	0	0.3	0.7

Table 8: Probability Transition Matrix, M, trying to predict the weather

	RR	RS	SR	SS
RR	0.36	0.24	0.24	0.16
RS	0.36	0.24	0.12	0.28
SR	0.36	0.24	0.24	0.16
SS	0.18	0.12	0.21	0.49

Table 9: Matrix M after 2 consecutive days (aka M^2)

SP §1.3

8. **Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains. Give reasons for your answers.**

(*a*)	**1**	**2**	**3**	**4**	**5**
1	0.4	0.3	0.3	0	0
2	0	0.5	0	0.5	0
3	0.5	0	0.5	0	0
4	0	0.5	0	0.5	0
5	0	0.3	0	0.3	0.4

(*b*)	**1**	**2**	**3**	**4**	**5**	**6**
1	0.1	0	0	0.4	0.5	0
2	0.1	0.2	0.2	0	0.5	0
3	0	0.1	0.3	0	0	0.6
4	0.1	0	0	0.9	0	0
5	0	0	0	0.4	0	0.6
5	0	0	0	0	0.5	0.5

(*c*)	**1**	**2**	**3**	**4**	**5**
1	0	0	0	0	1
2	0	0.2	0	0.8	0
3	0.1	0.2	0.3	0.4	0
4	0	0.6	0	0.4	0
5	0	0.3	0	0.3	0.4

(*d*)	**1**	**2**	**3**	**4**	**5**	**6**
1	0.8	0	0	0.2	0	0
2	0	0.5	0	0	0.5	0
3	0	0	0.3	0.4	0.3	0
4	0.1	0	0	0.9	0	0
5	0	0.2	0	0	0.8	0
5	0.7	0	0	0.3	0	0

(a)

Solution. By looking at the matrix, or drawing out the structure, it trivially follows that $\{2, 4\}$ is a closed, irreducible set. This implies that those states are recurrent. Every other state is transient.

(b)

Solution. By looking at the matrix, or drawing out the structure, it trivially follows that $\{1, 4, 5\}$ is a closed, irreducible set. This means that state 1,4,5 are recurrent. Because State 6 communicates with state 5 and vice versa, this means 6 is also recurrent. 2 and 3 are transient states.

(c)

Solution. By looking at the matrix, or drawing out the structure, it trivially follows that $\{1,5\},\{2,4\}$ are both closed, irreducible sets. This means States 1,2,4,5 are recurrent, and state 3 is transient.

(d)

Solution. By looking at the matrix, or drawing out the structure, it trivially follows that $\{1,4\},\{2,5\}$ are both closed, irreducible sets. This means States 1,2,4,5 are recurrent, and state 3 and 6 are transient.

(e)

Solution. By looking at the matrix, or drawing out the structure, it trivially follows that $\{2,4\},\{1\}$ are both closed, irreducible sets. This means States 1,2,4 are recurrent, and state 3 and 5 are transient.