Math 190: Mathematics of Sustainability
Emory University
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Homework # 5

Markov Chains: Definitions and Examples

SP §1.1

1. **A fair coin is tossed repeatedly with results $Y_0, Y_1, Y_2, \ldots$ that are 0 or 1 with probability 1/2 each. For $n \geqslant 1$ let $X_n = Y_n + Y_{n-1}$ be the number of 1's in the $(n-1)$th and nth tosses. Is X_n a Markov chain?**

2. **Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n.**

3. **We repeated roll two four sided dice with numbers 1, 2, 3, and 4 on them. Let Y_k be the sum on the kth roll, $S_n = Y_1 + \cdots + Y_n$ be the total of the first n rolls, and $X_n = S_n(\mod 6)$. Find the transition probability for X_n**

SP §1.2

4. **The 1990 census showed that 36% of the households in the District of Columbia were homeowners while the remainder were renters. During the next decade 6% of the homeowners became renters and 12% of the renters became homeowners. What percentage were homeowners in 2000? in 2010?**

5. **Consider a gambler's ruin chain with $n = 4$. That is, if $1 \leqslant i \leqslant 3$, $p(i, i+1) = 0.4$, and $p(i, i-1) = 0.6$, but the endpoints are absorbing states: $p(0, 0) = 1$ and $p(4, 4) = 1$. Compute $p^3(1, 4)$ and $p^3(1, 0)$.**

6. **A taxicab driver moves between the airport$A$ and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4.**

   (a) **Find the transition matrix for the chain.**

   (b) **Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.**

7. **Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$.**

   (a) **Compute its transition probability.**

   (b) **Compute the two-step transition probability.**

   (c) **What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday**

SP §1.3

8. **Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains. Give reasons for your answers.**

(a)	**1**	**2**	**3**	**4**	**5**
1	0.4	0.3	0.3	0	0
2	0	0.5	0	0.5	0
3	0.5	0	0.5	0	0
4	0	0.5	0	0.5	0
5	0	0.3	0	0.3	0.4

(b)	**1**	**2**	**3**	**4**	**5**	**6**
1	0.1	0	0	0.4	0.5	0
2	0.1	0.2	0.2	0	0.5	0
3	0	0.1	0.3	0	0	0.6
4	0.1	0	0	0.9	0	0
5	0	0	0	0.4	0	0.6
5	0	0	0	0	0.5	0.5

(c)	**1**	**2**	**3**	**4**	**5**
1	0	0	0	0	1
2	0	0.2	0	0.8	0
3	0.1	0.2	0.3	0.4	0
4	0	0.6	0	0.4	0
5	0	0.3	0	0.3	0.4

(d)	**1**	**2**	**3**	**4**	**5**	**6**
1	0.8	0	0	0.2	0	0
2	0	0.5	0	0	0.5	0
3	0	0	0.3	0.4	0.3	0
4	0.1	0	0	0.9	0	0
5	0	0.2	0	0	0.8	0
5	0.7	0	0	0.3	0	0